RAMSEY FOR COMPLETE GRAPHS WITH DROPPED CLIQUES

JONATHAN CHAPPELON, LUIS PEDRO MONTEJANO,
AND JORGE LUIS RAMÍREZ ALFONSÍN

Abstract. Let $K_{[k,t]}$ be the complete graph on k vertices from which a set of edges, induced by a clique of order t, has been dropped. In this note we give two explicit upper bounds for $R(K_{[k_1,t_1]},\dots,K_{[k_r,t_r]})$ (the smallest integer n such that for any r-edge coloring of K_n there always occurs a monochromatic $K_{[k_i,t_i]}$ for some i). Our first upper bound contains a classical one in the case when $k_1=\dots=k_r$ and $t_i=1$ for all i. The second one is obtained by introducing a new edge coloring called χ_r*-colorings*. We finally discuss a conjecture claiming, in particular, that our second upper bound improves the classical one in infinitely many cases.

1. Introduction

Let K_n be a complete graph and let $r \geqslant 2$ be an integer. A r-edge coloring of a graph is a surjection from $E(G)$ to $\{0,\dots,r-1\}$ (and thus each color class is not empty). Let $k \geqslant t \geqslant 1$ be positive integers. We denote by $K_{[k,t]}$ the complete graph on k vertices from which a set of edges, induced by a clique of order t, has been dropped, see Figure 1.

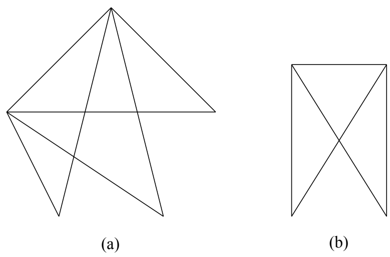

Figure 1. (a) $K_{[5,3]}$ and (b) $K_{[4,2]}$

Let $k_1,\dots,k_r$ and $t_1,\dots,t_r$ be positive integers with $k_i \geqslant t_i$ for all $i \in \{1,\dots,r\}$. Let $R([k_1,t_1],\dots,[k_r,t_r])$ be the smallest integer n such that for any r-edge coloring of K_n there always occurs a monochromatic $K_{[k_i,t_i]}$ for some i.
In the case when $k_i = t_i$ for some i, we set

$$R([k_1,t_1],\dots,[k_{i-1},t_{i-1}],[t_i,t_i],[k_{i+1},t_{i+1}],\dots,[k_r,t_r]) = t_i$$

since the set of all the edges of $K_{[t_i,t_i]}$ (which is empty) can always be colored with color i. We notice that the case $R([k_1,1],\dots,[k_r,1])$ is exactly the classical Ramsey number $r(k_1,\dots,k_r)$ (the smallest integer n such that for any r-edge coloring of K_n there always occurs a monochromatic K_{k_i} for some i). We refer the reader to the excellent survey [6] on Ramsey numbers for small values.

In this note, we investigate general upper bounds for $R([k_1,t_1],\dots,[k_r,t_r])$. In the next section we present a recursive formula that yields to an explicit general upper bound

Date: July 23, 2013.
2010 *Mathematics Subject Classification.* 05C55, 05D10.
Key words and phrases. Ramsey number, recursive formula.
The second author was supported by CONACYT.

(Theorem 2.2) generalizing the well-known explicit upper bound due to Graham and Rödl [4] (see equation 2). We also improve our explicit upper bound when $r = 2$ for certain values of k_i, t_i (Proposition 2.4).

In Section 3, we shall present another general explicit upper bound for $R([k_1, t_1], \ldots, [k_r, t_r])$ (Theorem 3.8) by introducing a new edge coloring called *χ_r-colorings*. We end by discussing a conjecture that is supported by graphical and numerical results.

2. The upper bound

The following recursive inequality is classical in Ramsey theory

$$
\begin{aligned}
(1) \qquad r(k_1, k_2, \ldots, k_r) \;&\leqslant r(k_1 - 1, k_2, \ldots, k_r) + r(k_1, k_2 - 1, \ldots, k_r) + \cdots + \\
&+ r(k_1, k_2, \ldots, k_r - 1) - (r - 2)
\end{aligned}
$$

In the same spirit, we have the following.

Lemma 2.1. *Let $r \geqslant 2$ and let $k_1, \ldots, k_r$ and $t_1, \ldots, t_r$ be positive integers with $k_i \geqslant t_i + 1 \geqslant 2$ for all i. Then,*

$$
\begin{aligned}
R([k_1, t_1], \ldots, [k_r, t_r]) \;\leqslant\; & R([k_1 - 1, t_1], [k_2, t_2], \ldots, [k_r, t_r]) \\
& + R([k_1, t_1], [k_2 - 1, t_2], \ldots, [k_r, t_r]) \\
& \vdots \\
& + R([k_1, t_1], [k_2, t_2], \ldots, [k_r - 1, t_r]) - (r - 2).
\end{aligned}
$$

Proof. Let us take any r-edge coloring of K_N with

$$N \geqslant R([k_1 - 1, t_1], [k_2, t_2], \ldots, [k_r, t_r]) + \cdots + R([k_1, t_1], [k_2, t_2], \ldots, [k_r - 1, t_r]) - (r - 2).$$

Let v a vertex of K_N and let $\Gamma_i(v)$ be the set of all vertices joined to v by an edge having color i for each $i = 1, \ldots, r$. We claim that there exists index $1 \leqslant i \leqslant r$ such that

$$\Gamma_i(v) \geqslant R([k_1, t_1], \ldots, [k_i - 1, t_i], \ldots, [k_r, t_r]).$$

Otherwise,

$$
\begin{aligned}
N - 1 = d(v) = \sum_{j=1}^{r} \Gamma_j(v) \;&\leqslant \sum_{j=1}^{r} (R([k_1, t_1], \ldots, [k_i - 1, t_i], \ldots, [k_r, t_r]) - 1) \\
&= \sum_{j=1}^{r} (R([k_1, t_1], \ldots, [k_i - 1, t_i], \ldots, [k_r, t_r]) - r \\
&\leqslant N + (r - 2) - r = N - 2
\end{aligned}
$$

which is a contradiction.

Now, suppose that $\Gamma_i(v) \geqslant R([k_1, t_1], \ldots, [k_i - 1, t_i], \ldots, [k_r, t_r])$ for an index i. By definition of $R([k_1, t_1], \ldots, [k_i - 1, t_i], \ldots, [k_r, t_r])$ we have that the complete graph induced by $\Gamma_i(v)$ contains either a subset of vertices inducing a copy $K_{[k_j, t_j]}$ having all edges with color j, for some $j \neq i$, and we are done or a subset of vertices inducing $K_{[k_i - 1, t_i]}$ having all edges with color i. Adding vertex v to $K_{[k_i - 1, t_i]}$ we obtain the desired copy of $K_{[k_i, t_i]}$ having all edges colored with color i. □

A similar recursive inequality has been treated in [7] in a more general setting (by considering a family of graphs intrinsically constructed via two operations *disjoin unions* and *joins*, see also [5] for the case $r = 2$). Although the latter could be used to obtain Lemma 2.1, the arguments used here give a different and a more straight forward proof. Moreover, our approach yield us to the following general upper bound for $R([k_1, t_1], \ldots, [k_r, t_r])$. The latter was not treated in [7] at all (suitable values/bounds needed to upper bound the recursion given in [7] for such general family seem to be very difficult to estimate).

Theorem 2.2. *Let $r \geqslant 2$ be a positive integer and let $k_1, \ldots, k_r$ and $t_1, \ldots, t_r$ be positive integers such that $k_i \geqslant t_i$ for all $i \in \{1, \ldots, r\}$. Then,*

$$R\left([k_1, t_1], \ldots, [k_r, t_r]\right) \leqslant \max_{1 \leqslant i \leqslant r}\{t_i\} \binom{k_1 + \cdots + k_r - (t_1 + \cdots + t_r)}{k_1 - t_1, k_2 - t_2, \ldots\ldots, k_r - t_r}$$

where $\binom{n_1+n_2+\cdots+n_r}{n_1,n_2,\ldots\ldots,n_r}$ is the multinomial coefficient defined by $\binom{n_1+n_2+\cdots+n_r}{n_1,n_2,\ldots\ldots,n_r} = \frac{(n_1+\cdots+n_r)!}{n_1!n_2!\cdots n_r!}$, for all nonnegative integers $n_1, \ldots, n_r$.

Proof. We suppose that $t_1, \ldots, t_r$ are fixed. We proceed by induction on $k_1 + \cdots + k_r$, using Lemma 2.1. In the case where $k_j = t_j$, for some $j \in \{1, \ldots, r\}$, we already know that

$$R\left([k_1, t_1], \ldots, [k_{j-1}, t_{j-1}], [t_j, t_j], [k_{j+1}, t_{j+1}], \ldots, [k_r, t_r]\right) = t_j,$$

and, since $k_i - t_i \geqslant 0$ for all i,

$$\binom{k_1 + \cdots + k_{i-1} + k_{i+1} + \cdots + k_r - (t_1 + \cdots + t_{i-1} + t_{i+1} + \cdots + t_r)}{k_1 - t_1, \ldots, k_{j-1} - t_{j-1}, 0, k_{j+1} - t_{j+1} \ldots\ldots, k_r - t_r} \geqslant 1.$$

Therefore

$$R\left([k_1, t_1], \ldots, [k_r, t_r]\right) = t_j \leqslant \max_{1 \leqslant i \leqslant r} t_i \binom{k_1 + \cdots + k_r - (t_1 + \cdots + t_r)}{k_1 - t_1, k_2 - t_2, \ldots\ldots, k_r - t_r}$$

in this case. Now, suppose that $k_i > t_i$ for all $i \in \{1, \ldots, r\}$. By Lemma 2.1 and by induction hypothesis, we obtain that

$$\begin{aligned} R([k_1, t_1], \ldots, [k_r, t_r]) \leqslant\ & R([k_1 - 1, t_1], [k_2, t_2], \ldots, [k_r, t_r]) \\ & + R([k_1, t_1], [k_2 - 1, t_2], \ldots, [k_r, t_r]) \\ & \vdots \\ & + R([k_1, t_1], [k_2, t_2], \ldots, [k_r - 1, t_r]) - (r - 2) \\ \leqslant\ & \max_{1 \leqslant i \leqslant r} t_i \left(\binom{k_1 + \cdots + k_r - (t_1 + \cdots + t_r) - 1}{k_1 - t_1 - 1, k_2 - t_2, \ldots\ldots, k_r - t_r} \right. \\ & + \binom{k_1 + \cdots + k_r - (t_1 + \cdots + t_r) - 1}{k_1 - t_1 - 1, k_2 - t_2 - 1, \ldots\ldots, k_r - t_r} \\ & \vdots \\ & \left. + \binom{k_1 + \cdots + k_r - (t_1 + \cdots + t_r) - 1}{k_1 - t_1 - 1, k_2 - t_2, \ldots\ldots, k_r - t_r - 1} \right) - (r - 2) \\ \leqslant\ & \max_{1 \leqslant i \leqslant r} t_i \binom{k_1 + \cdots + k_r - (t_1 + \cdots + t_r)}{k_1 - t_1, k_2 - t_2, \ldots\ldots, k_r - t_r}, \end{aligned}$$

since we have the following multinomial identity

$$\binom{n_1 + n_2 + \cdots + n_r}{n_1, n_2, \ldots\ldots, n_r} = \sum_{i=1}^{r} \binom{n_1 + n_2 + \cdots + n_r - 1}{n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_r}$$

for all positive integers $n_1, n_2, \ldots, n_r$. □

Let $R_r([k, t]) = R(\underbrace{[k, t], \ldots, [k, t]}_{r})$.

Corollary 2.3. *Let $k \geqslant t \geqslant 2$ and $r \geqslant 2$ be integers. Then,*

$$R_r([k,t]) \leqslant t\binom{r(k-t)}{k-t,\ldots,k-t}.$$

An immediate consequence of the above corollary (when $t = 1$) is the following classical upper bound due to Graham and Rödl [4] that was obtained by using (1).

$$R_r([k,1]) \leqslant \frac{(rk-r)!}{((k-1)!)^r}. \tag{2}$$

2.1. **Case** $r = 2$**.** When $r = 2$, it is the exact values of the recursive sequence generated from $u_{t,k} = u_{k,t} = t (= R_2([t,t]))$ for all $k \geqslant t$ and following the recursive identity $u_{k_1,k_2} = u_{k_1-1,k_2} + u_{k_1,k_2-1}$ for all $k_1, k_2 \geqslant t+1$.

We investigate with more detail the cases $R([s,2],[t,2])$ (resp. $R([s,2],[t,1])$), that is, the smallest integer n such that for any 2-edge coloring of K_n there always occurs a monochromatic $K_s - \{e\}$ or $K_t - \{e\}$ (resp. a monochromatic $K_s - \{e\}$ or K_t)). These cases have been extensively studied and values/bounds for specific s and t are known, see Table 1 obtained from [6].

	$K_3 \setminus \{e\}$	$K_4 \setminus \{e\}$	$K_5 \setminus \{e\}$	$K_6 \setminus \{e\}$	$K_7 \setminus \{e\}$	$K_8 \setminus \{e\}$	$K_9 \setminus \{e\}$	$K_{10} \setminus \{e\}$	$K_{11} \setminus \{e\}$
$K_3 \setminus \{e\}$	3	5	7	9	11	13	15	17	19
$K_4 \setminus \{e\}$	5	10	13	17	28	[29,38]	34	41	
$K_5 \setminus \{e\}$	7	13	22	[31,39]	[40,66]				
$K_6 \setminus \{e\}$	9	17	[31,39]	[45,70]	[59,135]				
$K_7 \setminus \{e\}$	13	28	[40,66]	[59,135]	251				
K_3	5	7	11	17	21	25	31	[37,39]	[42,47]
K_4	7	11	19	[27,34]	[37,52]	77	105	143	187
K_5	9	16	[30,34]	[43,67]	112	186	277	418	586
K_6	11	21	[37,53]	114	205	385	621	1035	1551
K_7	13	[28,31]	[51,84]	197	394	768	1339	2355	3766
K_8	15	42	123	306	659	1382	2562	4844	8223

TABLE 1. Known bounds and values of $R([s,2],[t,2])$ and $R([s,2],[t,1])$.

Proposition 2.4. *(a)* $R([3,2],[3,2]) = R([3,2],[2,1]) = 3$,

(b) $R([3,2],[k,2]) = 2k-3$ *and* $R([3,2],[k,1]) = 2k-1$,

(c) $R([4,2],[k,2]) \leqslant k^2-2k-39$ *for each* $k \geqslant 11$ *and*

(d) $R([4,2],[k,1]) \leqslant k^2-22$ *for each* $k \geqslant 9$.

Proof. (a) Let us show that $R([3,2],[3,2]) = 3$ (the proof for $R([3,2],[2,1]) = 3$, is analogous). For, we notice that $K_{[3,2]}$ is the graph consisting of three vertices, one of degree 2 and two of degree 1, and so $R([3,2],[3,2]) > 2$. Now, for any 2-coloring of the edges of K_3 there is always a vertex with two incident edges with the same color, giving the desired $K_{[3,2]}$.

(b) We first prove that $R([3,2],[k,2]) \leqslant 2k-3$. For, we iterate inequality of Lemma 2.1 obtaining

$$\begin{aligned} R([3,2],[k,2]) &\leqslant R([2,2],[k,2]) + R([3,2],[k-1,2]) \\ &= 2 + R([3,2],[k-1,2]) \\ &\leqslant 2 + R([2,2],[k-1,2]) + R([3,2],[k-2,2]) \\ &= 2 + 2 + R([3,2],[k-2,2]) \\ &\leqslant \cdots \leqslant \underbrace{2 + \cdots + 2}_{k-3} + R([3,2],[3,2]) \\ &= 2(k-3) + 3 = 2k-3. \end{aligned}$$

Now, we show that $R([3,2],[k,2]) > 2k-4$. For, take a perfect matching of $K_{2(k-2)}$. We color the edges belonging to the matching in red and all others in blue. We have neither a red $K_{[3,2]}$ red (since there are not vertex with two incident edges in red) nor a blue $K_{[k,2]}$ since any subset of k vertices forces to have at least two red edges.

(It can be proved that $R([3,2],[k,1]) = 2k-1$ by using completely similar arguments as above).

(c) It is known [3] that $R([4,2],[10,2]) = 41$. By using the latter and the recurrence of Lemma 2.1 we obtain

$$\begin{aligned} R([4,2],[k,2]) &\leqslant \sum_{i=11}^{k} R([3,2],[i,2]) + R([4,2],[10,2]) \\ &\leqslant \sum_{i=11}^{k} (2i-3) + 41 = (k-10)(k+11) - 3(k-10) + 41 = k^2 - 2k - 39. \end{aligned}$$

(d) It is known [1] that $R([4,2],[8,1]) = 42$. By using the latter and the recurrence of Lemma 2.1 we obtain

$$\begin{aligned} R([4,2],[k,1]) &\leqslant \sum_{i=9}^{k} R([3,2],[i,1]) + R([4,2],[8,1]) \\ &\leqslant \sum_{i=9}^{k} (2i-1) + 42 = (k-8)(k+9) - (k-8) + 41 = k^2 - 22. \end{aligned}$$

□

3. χ_r-Colorings

An r-edge coloring of K_n is said to be a *χ_r-coloring*, if there exists a labeling of $V(K_n)$ with $\{1,\ldots,n\}$ and a function $\phi : \{1,\ldots,n\} \to \{0,\ldots,r-1\}$ such that for all $1 \leqslant i < j \leqslant n$ the edge $\{i,j\}$ has color t if and only if $\phi(i) = t$.

Remark 3.1. (a) Notice that the value $\phi(n)$ do not play any role in the coloring.

(b) A *monochromatic* edge coloring (all edges have the same color $0 \leqslant t \leqslant r-1$) of K_n is a χ_r-coloring. Indeed, it is enough to take any vertex labeling and to set $\phi(i) = t$ for all i.

(c) There exist r-edge colorings of K_n that are not χ_r-coloring. For instance, it can be checked that for any labeling of $V(K_3)$ there is not a suitable function ϕ giving three different colors to the edges of K_3.

Exemple 3.2. A 2-coloring of K_3 with two edges of the same color and the third one with different color is a χ_2-coloring. Indeed, If the edges $\{1,2\}$ and $\{1,3\}$ are colored with color 0 and the edge $\{2,3\}$ with color 1 then we take $\phi(1) = 0, \phi(2) = 1$ and $\phi(3) = 1$.

Let $k \geqslant 1$ be an integer. Let $\chi_r(k)$ be the smallest integer n such that for any r-edge-coloring of K_N, $N \geqslant n$ there exist a clique of order k in which the induced r-edge coloring is a χ_r-coloring.

Remark 3.3. $\chi_r(k)$ always exists. Indeed, by Ramsey's theorem, for any r-edge coloring of K_N, $N \geqslant R_r(K_k)$ there exist a clique order k that is monochromatic which, by Remark 3.1 (b), is a χ_r-coloring.

We clearly have that $\chi_r(2) = 2$. For $\chi_r(3)$, we first notice that $\chi_r(3) = R_r([3,2])$ and that $K_{[3,2]}$ is a *star* $K_{1,2}$(a graph on three vertices, one of degree 2 and two of degree one). Now, Burr and Roberts [2] proved that

$$R(K_{1,q_1}, \ldots, K_{1,q_n}) = \sum_{j=1}^{n} q_j - n + \epsilon$$

where $\epsilon = 1$ if the number of even integers in the set $\{q_1, \ldots, q_n\}$ is even, $\epsilon = 2$ otherwise. Therefore, by applying the above formula when $q_i = 2$ for all i, we obtain

$$\chi_r(3) = \begin{cases} r+1 & \text{for } r \text{ even}, \\ r+2 & \text{for } r \text{ odd}. \end{cases} \tag{3}$$

Theorem 3.4. *Let $r \geqslant 2$ be a positive integer and let $k_1, \ldots, k_r$ and $t_1, \ldots, t_r$ be positive integers such that $k_i \geqslant t_i$ for all $i \in \{1, \ldots, r\}$. Then,*

$$R([k_1, t_1], \ldots, [k_r, t_r]) \leqslant \chi_r\left(\sum_{i=1}^{r}(k_i - t_i - 1) + 1 + \max_{1\leqslant i\leqslant r}\{t_i\}\right).$$

Proof. Consider a χ_r-coloring of $K_{\chi_r\left(\sum_{i=1}^{r}(k_i-t_i-1)+1+\max_{1\leqslant i\leqslant r}\{t_i\}\right)}$. Given the vertex labeling of the χ_r-coloring, we consider the complete graph K' induced by the vertices with labels $1, \ldots, \sum_{i=1}^{r}(k_i - t_i - 1) + 1$ (that is, we remove all the edges induced by the set of vertices T_1 with the $\max_{1\leqslant i\leqslant r}\{t_i\}$ largest labels). By the pigeonhole principle, there is a set T_2 of at least $k_i - t_i + 1 - 1$ vertices of K' with the same color for some i. Moreover, by definition of χ_r-coloring any edge $\{v_1, v_2\}$ with $v_1 \in T_1$ and $v_2 \in T_2$ has color i, giving the desired monochromatic $K_{[k_i,t_i]}$. □

The following result is an immediate consequence of Theorem 3.4.

Corollary 3.5. *Let $r, k \geqslant 2$ be integers. Then,*

$$R_r([k,1]) \leqslant \chi_r(r(k-2)+2) \text{ and } R_r([k,2]) \leqslant \chi_r(r(k-3)+3).$$

Proposition 3.6. *Let $r, k \geqslant 2$ be integers. Then,*

$$\chi_r(k) \leqslant r\chi_r(k-1) - r + 2.$$

Proof. Consider a r-edge coloring of $K_{r\chi_r(k-1)-r+2}$ and let u be a vertex. Since $d(u) = r\chi_r(k-1) - r + 1$ then there are at least $\left\lceil \frac{r\chi_r(k-1)-r+1}{r} \right\rceil = \chi_r(k-1)$ set of edges with the same color all incident to u. Now, by definition of $\chi_r(k-1)$, there is a clique H of order $k-1$ which edge coloring is a χ_r- coloring. So, there is a labeling π of $V(H)$, $|V(H)| = k$ and a function ϕ giving such coloring. We claim that the r-edge coloring of the clique $H' = H \cup u$ is a χ_r-coloring. Indeed, by taking the label $\pi'(i) = \pi(i) + 1$ for all vertex $i \neq u$ and $\pi'(u) = 1$ and the function $\phi'(1) = 1$ and $\phi'(i) = \phi(i-1)$ for each $i = 2, \ldots, k$. □

Proposition 3.7. *Let $r, k \geqslant 2$ be integers. Then,*

$$\chi_r(k) \leqslant g(k,r) = \begin{cases} r^{k-2} + r^{k-3} + \cdots + r^2 + r + 2 = \dfrac{r^{k-1}-1}{r-1} + 1 & \text{for } r \text{ odd,} \\ r^{k-2} + r^{k-4} + r^{k-5} + \cdots + r^2 + r + 2 = \dfrac{r^{k-3}-1}{r-1} + r^{k-2} + 1 & \text{for } r \text{ even.} \end{cases}$$

Proof. By equality (3) and by successive applications of Proposition 3.6. □

Theorem 3.8. *Let $r \geqslant 2$ be a positive integer and let $k_1, \ldots, k_r$ and $t_1, \ldots, t_r$ be positive integers such that $k_i \geqslant t_i$ for all $i \in \{1, \ldots, r\}$. Then,*

$$R([k_1, t_1], \ldots, [k_r, t_r]) \leqslant g(k, r)$$

where

$$k := \sum_{i=1}^{r} (k_i - t_i - 1) + 1 + \max_{1 \leqslant i \leqslant r} \{t_i\}.$$

Proof. By Theorem 3.4 and Proposition 3.7. □

We believe that the above upper bound for $R_r([k,1])$ is smaller than the one given by Corollary 2.3 (see equation (2)) for some values of k.

Conjecture 3.9. Let $r \geqslant 3$ be an integer. Then, for all $3 \leqslant k \leqslant r^{3/2} + r - 1$

$$g((r(k-2)+2, r) < \binom{r(k-1)}{k-1, k-1, \ldots\ldots, k-1} = \frac{(rk-r)!}{((k-1)!)^r}.$$

We have checked the validity of the above conjecture for all $3 \leqslant r \leqslant 150$ by computer calculations. Conjecture 3.9 is also supported graphically, by considering the continual behaviour of

$$f(k,r) = g((r(k-2)+2, r) - \frac{(rk-r)!}{((k-1)!)^r}.$$

To see that, we may use the fact that $\Gamma(z+1) = z!$ when z is a nonnegative integer, obtaining

$$f(k,r) = g((r(k-2)+2, r) - \frac{\Gamma(r(k-1)+1)}{\Gamma^r(k)}$$

where $\Gamma(z)$ is the well-known *gamma* function[1], see Figure 2.
We have also checked (by computer) that for each $3 \leqslant r \leqslant 150$ there is an interval I_r (increasing as r is growing) such that for each $k \geqslant 3, k \in I_r$ the function $g(r(k-3)+3, r)$ (resp. $g(r(k-4)+4, r)$) is a smaller upper bound for $R_r([k,2])$ (resp. for $R_r([k,3])$) than the corresponding ones obtained from Corollary 2.3. In view of the latter, we pose the following

Question 3.10. Let $t \geqslant 1$ and $r \geqslant 3$ be integers. Is there a function $c(r)$ such that for all $3 \leqslant k \leqslant c(r)$

$$g(r(k-t)+t, r) < t\binom{r(k-t)}{k-t, k-t, \ldots\ldots, k-t} \ ?$$

[1]The gamma function is defined as $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt$ for any $z \in \mathbb{C}$ with $Re(z) > 0$. Moreover, $\Gamma(z+1) = z!$ when z is a nonnegative integer.

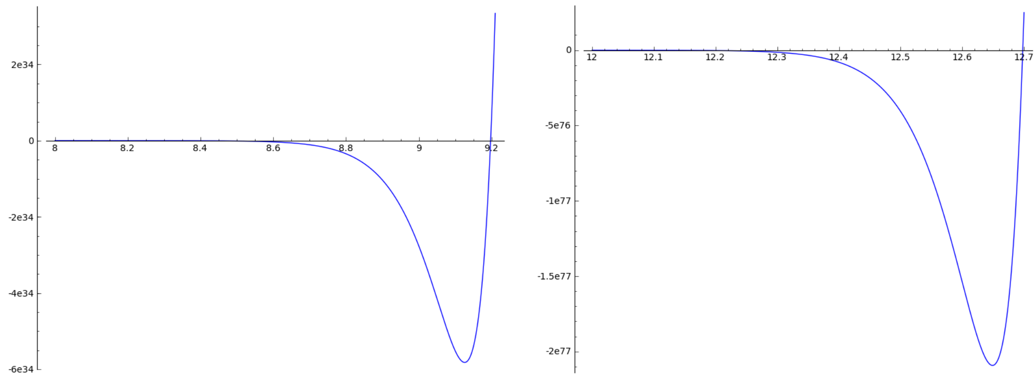

Figure 2. Behaviours of $f(4,k)$ avec $8 \leqslant k < 10$ (left) and $f(5,k)$ avec $12 \leqslant k < 13$ (right). We notice that due to the scaling used in the figures (in order to plot the minimum) the function f seems very close to zero but in fact it is very far apart, $f(4,8) \leqslant -1,8 \times 10^{29}$ for the left one and $f(5,12) \leqslant -5,7 \times 10^{72}$ for the right one.

References

[1] L. Boza, Nuevas Cotas Superiores de Algunos Números de Ramsey del Tipo $r(K_m, K_n - e)$, in *Proceedings of the VII Jornada de Matemática Discreta y Algoritmica, JMDA 2010*, Castro Urdiales, Spain, July 2010.

[2] S.A. Burr, J.A. Roberts, On Ramsey numbers for stars, *Utilitas Math.*, **4** (1973), 217–220.

[3] V. Chvátal, F. Harary, Generalized Ramsey theory for graphs. II Small diagonal numbers, *Proc. Amer. Math. Soc.* **32** (1972), 389-394.

[4] R. Graham and V. Rödl, Numbers in Ramsey theory, *Surveys in Combinatorics* 1987, 123, London Mathematics Society Lecture Note Series (1987) 111–153.

[5] Y.R. Huang, K. Zhang, New upper bounds for Ramsey numbers, *European J. Combin.* **19**(3) (1998), 391-394.

[6] S.P. Radziszowski, Small Ramsey numbers, *Electron. J. Combin.* **1** (1994), Dynamic Survey 1, 30 pp (electronic).

[7] L. Shi, K. Zhang, A bound for multicolor Ramsey numbers, *Discrete Math.* **226**(1-3) (2001), 419-421.

Université Montpellier 2, Institut de Mathématiques et de Modélisation de Montpellier, Case Courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex 05, France.
E-mail address: jonathan.chappelon@um2.fr
E-mail address: luispedro81@yahoo.com.mx
E-mail address: jramirez@math.univ-montp2.fr