RESTRICTING TORAL SUPERCUSPIDAL REPRESENTATIONS TO THE DERIVED GROUP, AND APPLICATIONS TO THE QUATERNION DIVISION ALGEBRA

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ABSTRACT. We determine the decomposition of the restriction of a length-one toral supercuspidal representation of a connected reductive group to algebraic derived subgroup. As an application, we determine the smooth dual of the unit group ${\mathcal{O}_D}^\times$ of a quaternion algebra D over a p-adic field F, for $p \neq 2$. We determine the branching rules for the restriction of representations of $D^\times \supset {\mathcal{O}_D}^\times \supset D^1$.

1. INTRODUCTION

Let $\mathbb{G}$ be a connected reductive algebraic group defined over a local non-archimedean field F, and set $G = \mathbb{G}(F)$. Under certain tameness assumptions, all irreducible supercuspidal representations may be constructed in a uniform way, starting from generic cuspidal G-data [1, 14, 10]. J. Hakim and F. Murnaghan [7] determined the equivalence classes of G-data which give rise to isomorphic supercuspidal representations. In this paper we consider the subset of *generic toral cuspidal G-data of length one* (abbreviated: G-data) and their corresponding supercuspidal representations.

Let $\mathbb{G}^1$ denote the derived group of $\mathbb{G}$; then this is a connected semisimple group over F. Set $G^1 = \mathbb{G}^1(F)$ and note that this may be strictly larger than the commutator subgroup of G^1. Restricting a G-datum Ψ to G^1 produces a datum Ψ^1 for G^1; in Proposition 2.2 we show that Ψ^1 meets the criteria of G^1-datum, and that $\pi(\Psi^1$ occurs in the restriction to G^1 of $\pi(\Psi)$. We prove a compatibility result in Theorem 5.1, whence we deduce the full decomposition of the restriction of $\pi(\Psi)$ into irreducible (supercuspidal) representations of G^1 in Corollary 5.2.

Our results hold modulo certain hypotheses. The classification of (toral) supercuspidal representations by G-data in [14, 10, 7] holds when G is split over a tamely ramified extension of F and p is sufficiently large (measured relative to the type of the root datum of $\mathbb{G}$). For example, the simple criterion of genericity that we apply requires p not to be a torsion prime for the dual root daum of $\mathbb{G}$. The restriction of a representation of G to G^1 is a simple direct sum when, for example, ZG^1 is an open

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subgroup of G of finite index; this always holds if F is of characteristic zero. Finally, we restrict ourselves to those G-data containing a quasi-character χ of G which has the property that $\mathrm{Res}_{G^1}\chi = 1$.

The restriction to toral supercuspidal representations here sidesteps the need to consider the branching rules of depth-zero supercuspidal representations to the derived group, or equivalently, of cuspidal representations of Lie groups of finite type. On the other hand, it is expected that generalizing to G-data of length greater than one is possible, using the results in [7] as here.

As an application of these results, we consider the group $D^\times$, for D a quaternion algebra over a local non-archimedean field F of odd residual characteristic. The representation theory of $D^\times$ is well-known, having been determined by L. Corwin and R. Howe in [4, 5, 8]. Its algebraic derived group, which coincides with its commutator subgroup, is D^1, the subgroup of elements of reduced norm 1. The representation theory of D^1 is also well-known; see for example [11]. We give the branching rules for the restriction of representations of $D^\times$ to D^1 in Section 7.

What is more interesting is the representation theory of the maximal compact open subgroup $\mathcal{O}_D^\times$ of $D^\times$, which coincides with the invertible elements of its integer ring of D. This is not an algebraic group, and as such, the methods of the classification of [1, 14] do not apply. Nevertheless, using in part the branching rules for $D^\times$ to D^1 established above, we determine the full representation theory of $\mathcal{O}_D^\times$. Furthermore, in Section 8 we give a classification of their representations by equivalence classes of $\mathcal{O}_D^\times$-data, in analogy with the classification for $D^\times$ and D^1. This is a step towards the greater project of determining the representation theory of groups like $\mathcal{O}_D^\times$, which are algebraic groups over local rings.

This paper is organized as follows. We set our notation and recall the notion of genericity for positive-depth quasi-characters of tori in Section 2, where we relate these notions for G and G^1. We discuss a key ingredient of the construction, the Heisenberg-Weil lift, in Section 3, following [7], and prove Proposition 3.3, which is key to relating different G^1-data in later sections.

In Section 4 we recall the classification of toral supercuspidal representations of length one, following [14, 7]. We prove some additional properties of this parametrization in Proposition 4.4. Section 5 gives the branching rules for the restriction of toral supercuspidal representations of length one of G to G^1, modulo hypotheses discussed above.

We then turn to the case of $G = D^\times$. We recall known facts about G and G^1, including their representation theory, and prove some key technical results. In Section 7 we apply the preceding to determine the branching rules of the restriction of each representation of $D^\times$ to each of D^1 and $\mathcal{O}_D^\times$. We use these results to give a classification of the irreducible representations of $\mathcal{O}_D^\times$, up to equivalence, in Section 8.

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2. Notation and Genericity

Let F be a local norarchimedean field with residue field $\mathfrak{f}$ of residual characteristic p, with integer ring $\mathcal{O}_F$ and prime ideal $\mathcal{P}_F$ with uniformizer ϖ. We fix a character ψ of F which is trivial on $\mathcal{P}_F$ but nontrivial on $\mathcal{O}_F$.

Let $\mathbb{G}$ be a connected reductive group defined and tamely ramified over F. Denote by $\mathbb{G}^1 = [\mathbb{G},\mathbb{G}]$ its derived group and set $G = \mathbb{G}(F)$, $G^1 = \mathbb{G}^1(F)$. Let Z denote the center of G.

We assume that p is sufficiently large for: the existence of generic elements in the Lie algebra (p must not be bad for G [14, §7]), the decomposition of the Lie algebra of G in the proof of Proposition 2.2 ($p > k(\mathbb{G})$, the order of the kernel of the central isogeny $Z(\mathbb{G}) \times \mathbb{G}^1 \to \mathbb{G}$); the work with the Heisenberg-Weil lift ($p > 2$ [7, §2.3]); and the construction of positive-depth toral supercuspidal representations to apply ($\mathbb{G}$ split over a tamely ramified extension of F). We refer the reader to the the excellent discussion in [2, §1]. For the case $G = D^\times$ considered starting in Section 6, $p > 2$ is sufficient.

To each $x \in \mathcal{B}^{red}(\mathbb{G}, F) = \mathcal{B}(\mathbb{G}^1, F)$ and $r \in \mathbb{R}_{\geq 0}$ we associate the corresponding Moy-Prasad filtration subgroups $G_{x,r}$ and $G_{x,r+}$ as in [12]. In particular these give well-defined filtrations T_r of $T = \mathbb{T}(F)$ for any maximal torus $\mathbb{T}$ of $\mathbb{G}$ and, for any extension field E over which $\mathbb{T}$ is split and x in its corresponding apartment, filtrations $\mathbb{G}_\alpha(E)_{x,r}$ of each root subgroup $\mathbb{G}_\alpha(E)$ of $\mathbb{G}$ corresponding to $(\mathbb{G},\mathbb{T})$. We have corresponding decompositions at the level of Lie algebras, and for the dual of the Lie algebra. We refer the reader to [7, §2.5], for example, for a summary of the many useful properties of these filtrations.

Recall that the *depth* of a representation ρ of G is defined to be the least $r \in \mathbb{R}_{\geq 0}$ such that for some $x \in \mathcal{B}^{red}(\mathbb{G}, F)$, ρ contains vectors invariant under $G_{x,r+}$.

Let T be a maximal torus of G with Lie algebra $\mathfrak{t}$. For $r > 0$ we have an isomorphism $e\colon \mathfrak{t}_r/\mathfrak{t}_{r+} \to T_r/T_{r+}$ and furthermore any character of $\mathfrak{t}_r/\mathfrak{t}_{r+}$ is given by $X \mapsto \psi(\langle X^*, X\rangle)$ for some $X^* \in \mathfrak{t}^*_{-r}$.

Choose an extension field E of F over which T splits, and let $\Phi = \Phi(\mathbb{G},\mathbb{T},E)$ be the corresponding root system. For each $\alpha \in \Phi$, the coroot $\alpha^\vee\colon \mathbb{G}_m \to \mathbb{T}$ is defined over E and has linearization at 1 the element $H_\alpha = d\alpha^\vee(1) \in \mathfrak{t}(E)_0$. Thus for any X^* as above, one has $\mathrm{val}_E(\langle X^*, H_\alpha\rangle) \geq -r$.

Definition 2.1. With notation as above, an element $X^* \in \mathfrak{t}(F)^*_{-r}$ is $\mathbb{G}(F)$-generic of depth $-r$ if for each $\alpha \in \Phi$, $\mathrm{val}_E(\langle X^*, H_\alpha\rangle) = -r$.

This definition is taken from [14, §8]; it is the analogy on the dual of the Lie algebra to the notion of a *good element* defined in [1].

Given a character ϕ of T of positive depth r, it factors to a representation of $T_r/T_{r+} \cong \mathfrak{t}_r/\mathfrak{t}_{r+}$, where it is realized as

$$\phi(e(X)) = \Psi(\langle X^*, X\rangle)$$

for some $X^* \in \mathfrak{t}^*_{-r}$; we say ϕ is realized by X^*. The character ϕ is called G*-generic of depth* r if X^* is G-generic of depth $-r$. The following proposition could in most cases be deduced from [2, Lemma 5.9] via the intermediary of good elements, but is easy to prove directly here.

Proposition 2.2. *Let* $\mathbb{T}$ *be a maximal torus of* $\mathbb{G}$ *and* $T = \mathbb{T}(F)$*. Then a character* ϕ *of* T *is* G*-generic of depth* r *if and only if its restriction to* $T \cap G^1$ *is* G^1*-generic of depth* r*.*

Proof. Write $S = T \cap G^1$ and Z for the center of G. We have $S_r = T_r \cap G^1$. Denote the Lie algebras by the corresponding letter $\mathfrak{g}, \mathfrak{g}^1, \mathfrak{z}, \mathfrak{t}, \mathfrak{s}$. By [2, Proposition 3.1] we have $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{s}$ and $\mathfrak{s}_r = \mathfrak{t}_r \cap \mathfrak{g}^1$. Identify $\mathfrak{s}^*$ with the set of $X^* \in \mathfrak{t}^*$ which are trivial on $\mathfrak{z}$, and similarly for $\mathfrak{z}^*$; then we have a T-invariant decomposition $\mathfrak{t}^* = \mathfrak{z}^* \oplus \mathfrak{s}^*$.

Suppose ϕ is a character of T of depth r and let $X^* \in \mathfrak{t}^*_{-r}$ realize ϕ on T_r. Decompose $X^* = Z^* + Y^*$ with $Z \in \mathfrak{z}^*_{-r}$ and $Y^* \in \mathfrak{s}^*_{-r}$; then $\mathrm{Res}_{S_r}\phi$ is realized by Y^*. Let E be a splitting field of T; then as $span_E\{H_\alpha \mid \alpha \in \Phi\} = \mathfrak{s}(E)$, it follows that X^* is G-generic of depth $-r$ if and only if Y^* is G^1-generic of depth $-r$. □

3. On Heisenberg p-groups and Weil representations

We summarize some essential components in the construction of supercuspidal representations from [7, §2.3].

Let $(W, \langle,\rangle)$ be a finite-dimensional symplectic vector space over $\mathbb{F}_p$. Endow the set $W \times \mathbb{F}_p$ with the group operation $(w, z)(w', z') = (w + w', z + z' + \frac{1}{2}\langle w, w'\rangle)$, and denote the resulting Heisenberg group $W^\sharp$ or $W \boxtimes \mathbb{F}_p$.

This Heisenberg group carries a natural action of $Sp(W)$. For any choice of central character, a corresponding Heisenberg representation τ of $W^\sharp$ extends to a representation $\widehat{\tau} = (\tau_S, \tau)$ of the group $Sp(W) \ltimes W^\sharp$, called the Heisenberg-Weil lift of τ [7, Definition 2.17]. This extension is unique in all but one case (which occurs only if $p = 3$); in that case, a particular extension has been designated [7, §2.4].

An abstract p-Heisenberg group is a group H which is isomorphic to some $W^\sharp$. In this case its center Z_H is a cyclic group of order p. Fix a nontrivial character ϕ of Z_H; then ϕ defines an isomorphism of Z_H with $\mu_p \subset \mathbb{C}^\times$. We fix the isomorphism $\kappa\colon \mu_p \to \mathbb{F}_p$ given by $\kappa(e^{2\pi i/p}) = 1$. In this way we can also recover a choice of W from H and ϕ: set $W = H/Z_H$; then the pairing $\langle a, b\rangle := \kappa(\phi(aba^{-1}b^{-1}))$ defines the structure of a symplectic vector space on W, such that H is isomorphic with $W^\sharp$.

Definition 3.1. ([7, Definition 2.29]) For fixed ϕ and $W = H/Z_H$, an isomorphism $\nu\colon H \to W^\sharp$ is called *special* if the following diagram commutes:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Z_H & \longrightarrow & H & \longrightarrow & W & \longrightarrow & 1 \\
 & & {\scriptstyle \kappa\circ\phi}\downarrow \cong & & \downarrow{\scriptstyle \nu} & & \| & & \\
1 & \longrightarrow & \mathbb{F}_p & \longrightarrow & W^\sharp & \longrightarrow & W & \longrightarrow & 1
\end{array}$$

Note that a special homomorphism ν defines an action of $Sp(W)$ on H; we denote the resulting semidirect product $Sp(W)\ltimes_\nu H$. Then $1\times\nu\colon Sp(W)\ltimes_\nu H \to Sp(W)\ltimes W^\sharp$ is an isomorphism.

Let $Sp(H)$ denote the group of automorphisms of H which act by the identity on Z_H. Then we have a natural inclusion of $Sp(W)$ into $Sp(W^\sharp)$.

Definition 3.2. ([7, Definition 3.17]) Let T be a group equipped with a homomorphism $f'\colon T \to Sp(H)$. The special isomorphism ν is *relevant* for f' if the image of the map

$$\begin{aligned} T &\to Sp(W^\sharp) \\ t &\mapsto \nu \circ f'(t) \circ \nu^{-1} \end{aligned}$$

lies in the subgroup $Sp(W)$. In this case we write f_ν for the induced homomorphism $f_\nu\colon T \to Sp(W)$.

In other words, ν is relevant for f' if and only if $f_\nu \times 1\colon T \ltimes H \to Sp(W) \ltimes_\nu H$ is a group homomorphism.

Fix a Heisenberg representation τ of $W^\sharp$ of central character κ^{-1}, and denote by $\widehat{\tau} = (\tau_S, \tau)$ its Heisenberg-Weil lift to a representation of $Sp(W)\ltimes W^\sharp$. If $\nu\colon H \to W^\sharp$ is a special isomorphism, relevant for $f'\colon T \to Sp(H)$, then it induces a homomorphism $(f_\nu, \nu)\colon T \ltimes H \to Sp(W) \ltimes W^\sharp$. Then we may pull back the Heisenberg-Weil representation to define a representation

$$\widetilde{\phi}^\nu(t,h) = \tau_S(f_\nu(t))\tau(\nu(h))$$

of $T \ltimes H$, where here ϕ refers to the central character of H fixed in the definition of ν. By [7, Lemma 3.21], the isomorphism class of $\widetilde{\phi}^\nu$ is independent of the choice of special isomorphism relevant for f'.

Proposition 3.3. *For $i \in \{1,2\}$ let H_i be a Heisenberg group, T_i a group, $f_i'\colon T_i \to Sp(H_i)$ a homomorphism and $\nu_i\colon H_i \to W_i^\sharp$ a special isomorphism which is relevant for f_i'. Suppose we have a group isomorphism $\alpha\colon H_1 \to H_2$, a symplectic isomorphism $\beta\colon W_1 \to W_2$, and a homomorphism $\delta\colon T_1 \to T_2$ such that the following diagrams commute:*

$$\begin{array}{ccc}
T_1 & \xrightarrow{(f_1)_{\nu_1}} & Sp(W_1) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle inn(\beta)} \\
T_2 & \xrightarrow{(f_2)_{\nu_2}} & Sp(W_2)
\end{array}
\qquad\qquad
\begin{array}{ccc}
H_1 & \xrightarrow{\nu_1} & W_1^\sharp \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta\times id} \\
H_2 & \xrightarrow{\nu_2} & W_2^\sharp.
\end{array}$$

Let (τ_2, V) be a Heisenberg representation of $W_2^\sharp$ with central character ζ. Then $\tau_1 = \tau_2 \circ (\beta \times id)$ is a Heisenberg representation of $W_1^\sharp$ on the space V with the same central character, and the following diagram commutes:

$$\begin{array}{ccccccc}
T_1 \ltimes H_1 & \xrightarrow{(f_1)_{\nu_1}\times id} & Sp(W_1) \ltimes_{\nu_1} H_1 & \xrightarrow{id\times\nu_1} & Sp(W_1) \ltimes W_1^\sharp & \xrightarrow{(\tau_1)_S \ltimes \tau_1} & GL(V) \\
\downarrow{\scriptstyle \delta\times\alpha} & & \downarrow{\scriptstyle inn(\beta)\times\alpha} & & \downarrow{\scriptstyle inn(\beta)\times(\beta\times id)} & & \Vert \\
T_2 \ltimes H_2 & \xrightarrow{(f_2)_{\nu_2}\times id} & Sp(W_2) \ltimes_{\nu_2} H_2 & \xrightarrow{id\times\nu_2} & Sp(W_2) \ltimes W_2^\sharp & \xrightarrow{(\tau_2)_S \ltimes \tau_2} & GL(V)
\end{array}$$

Proof. Since ν_i is relevant for f_i, the commutativity of the first square follows from that of the first diagram of the hypothesis. Similarly, since ν_i is a special isomorphism, the commutativity of the second square follows from that of the second diagram of the hypothesis. When the Heisenberg-Weil extension is unique, the commutativity of the third diagram is immediate. In the remaining case, one verifies that the choice of extension $(\tau_i)_S$ made explicitly in [7, §2.4] is compatible with the change of base β. □

4. Toral supercuspidal representations of length one

We summarize the construction of irreducible supercuspidal representations of positive depth arising from toral generic G-data of length one. We follow the presentation in [7]. When G is of rank one (over a separable closure) and p is sufficiently large, all irreducible supercuspidal representations of G of positive depth arise either in this way, or else as a twist by a positive-depth character of G of a depth-zero representation. More generally, this is true of any connected reductive group whose longest tamely ramified twisted Levi sequence (in the sense of [14, §2]) has two factors.

4.1. **The datum.**

A *generic toral G-datum of length one* (abbreviated: G-datum) consists of: $T = \mathbb{T}(F)$, where $\mathbb{T}$ is a minisotropic maximal torus of $\mathbb{G}$, defined over F; a point $y \in \mathcal{B}^{red}(G, F) \cap \mathcal{A}(G, T, E)$, where E is a splitting field of $\mathbb{T}$; a G-generic quasi-character ϕ of T of positive depth r; and a quasi-character χ of G which is either trivial or else of depth $\tilde{r} \geq r$.

Remark 4.1. In [14], the construction depends on the choice of $y \in \mathcal{B}(G, F)$, but by [7, Remark 3.10] we deduce that in the toral case it depends only on the image of y in $\mathcal{B}^{red}(G, F)$, which in turn is uniquely determined by the minisotropic torus T.

We abbreviate such a datum as $\Psi = (T, y, \phi, r, \chi)$. For $g \in G$ we set ${}^g\Psi = ({}^gT, g \cdot y, {}^g\phi, r, \chi)$, where ${}^gT := gTg^{-1}$ and ${}^g\phi$ is the corresponding representation of ${}^gT$.

4.2. **The construction of $\widetilde{\rho}$.**

The main step is the construction of a representation $\widetilde{\rho}$ of $TG_{y,r/2}$ from the subset (T, y, ϕ, r) of the G-datum. We summarize it here, primarily following the detailed presentation in [7, §2.3 and 3.3].

Let E be a splitting field of T and set $\Phi = \Phi(\mathbb{G}, \mathbb{T}, E)$. We consider y as an element of $\mathcal{B}^{red}(\mathbb{G}, E)$. Define

$$J(E) = \langle \mathbb{T}(E)_r, \mathbb{G}_\alpha(E)_{y,s} \mid \alpha \in \Phi \rangle$$

and

$$J_+(E) = \langle \mathbb{T}(E)_r, \mathbb{G}_\alpha(E)_{y,s+} \mid \alpha \in \Phi \rangle.$$

Note that $\mathbb{T}(E)J(E) = \mathbb{T}(E)\mathbb{G}(E)_{y,s}$ and $\mathbb{T}(E)J_+(E) = \mathbb{T}(E)\mathbb{G}(E)_{y,s+}$.

The character ϕ of T is realized on $T_r/T_{r+} \cong \mathfrak{t}_r/\mathfrak{t}_{r+}$ by an element $X^* \in \mathfrak{t}^*_{-r}$, via the fixed additive character Ψ of F. Choose an extension of Ψ to E; then extending X^* to a linear functional on $\mathfrak{t}(E)$ similarly defines a character ϕ_E of $\mathbb{T}(E)_r/\mathbb{T}(E)_{r+}$, whose restriction to T_r moreover coincides with ϕ. We extend ϕ_E trivially across the groups $\mathbb{G}_\alpha(E)_{y,s+}$, $\alpha \in \Phi$, to produce the character $\widehat{\phi_E}$ of $J_+(E)$.

If $J(E) = J_+(E)$ then ϕ_E and $\widehat{\phi_E}$ together extend to a unique character of $\mathbb{T}(E)J(E)$, whose restriction to $TG_{y,s}$ we denote $\widehat{\phi}$. We then define

$$\widetilde{\rho} = \widehat{\phi}.$$

Note that in this case, one can also define $\widetilde{\rho}$ without the use of a field extension; see for example [14, §4]. Moreover, $\mathrm{Res}_T(\widetilde{\rho}) = \phi$.

Now suppose that $J(E) \neq J_+(E)$. Set $N(E) = \ker(\widehat{\phi_E})$. As $T_r/T_{r+} \cong \mathfrak{t}_r/\mathfrak{t}_{r+}$ is the additive group of a vector space over the residue field of E, which has characteristic p, the index of $N(E)$ in $J_+(E)$ is p. One verifies that $H(E) = J(E)/N(E)$ is an abstract Heisenberg group over $\mathbb{F}_p$ with center $Z_{H(E)} = J_+(E)/N(E)$.

We fix the central character $\widehat{\phi_E}$. Conjugation by elements of $\mathbb{T}(E)$ defines a homomorphism $f'_E \colon \mathbb{T}(E) \to Sp(H(E))$. The following construction produces a special isomorphism $\nu_E \colon H(E) \to W(E)^\sharp$, relevant for f'_E [7].

Since $W(E) \cong \oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha(E)_{y,s}/\mathfrak{g}_\alpha(E)_{y,s+}$, we may choose a polarization $W(+) \oplus W(-)$ of $W(E)$ where $W(\pm)$ is spanned by the positive (respectively, negative) root spaces. This lifts to a splitting of $H(E)$ as well, in the sense that each $g \in H(E)$ can be written in the form $g = g_+g_-g_0$ with $g_\pm Z_{H(E)} \in W(\pm)$ and $g_0 \in Z_{H(E)}$, and the map

$$\mu(g) = \kappa(\widehat{\phi_E}(g_0)) + \frac{1}{2}\langle g_+, g_- \rangle$$

is well-defined. Then $\nu_E(h) = (hZ_{H(E)}, \mu(h))$ is the isomorphism sought.

Finally, let J, J_+ and N denote the intersections with G of the corresponding groups over E. Set $H = J/N$, $W = J/J_+$ and $Z_H = J_+/N \cong Z_{H(E)}$. Then the restriction ν of ν_E to H also defines a special isomorphism with $W^\sharp$, relevant for the map $f' \colon T \to Sp(H)$ induced by conjugation. Let $f_\nu \colon T \to Sp(W)$ denote the induced homomorphism.

The group T also acts directly on $W = J/J_+$ by conjugation, inducing a map $f\colon T \to Sp(W)$. By [7, Lemma 3.18], we have $f = f_\nu$, implying in particular that this map is independent of the choice of ν.

Let τ denote a Heisenberg representation of $W^\sharp$ with central character κ^{-1}, and let $\widehat{\tau} = (\tau_S, \tau)$ denote its Heisenberg-Weil lift to $Sp(W) \ltimes W^\sharp$. Then we have the pullback representation of $T \ltimes J$, given on $t \in T$ and $j \in J$ by

$$\widetilde{\phi}(t,j) = \tau_S(f(t))\tau(\nu(j)).$$

We furthermore set $\widetilde{\rho}(tj) = \phi(t)\tilde{\phi}(t,j)$; this is well-defined as is the representation of $TJ = TG_{y,s}$ we sought.

Note that by [14, Theorem 11.5], $\mathrm{Res}_{T_{0+}}\widetilde{\rho}$ is ϕ-isotypic; but in general this is not true of $\mathrm{Res}_{T_0}\widetilde{\rho}$ due to the presence of the term $\tau_S(f(t))$.

4.3. **The representation $\pi_G(\Psi)$.** Let $\Psi = (T, y, \phi, r, \chi)$ be a G-datum and set $s = r/2$. Construct the representation $\widetilde{\rho}$ of $TG_{y,s}$ from the subset (T, y, ϕ, r) as above. Now define

$$\rho_G(\Psi) = \chi\widetilde{\rho}$$

which is again a representation of $TG_{y,s}$. The following is a special case of results in [1, 14].

Theorem 4.2. *The representation*

$$\pi_G(\Psi) = c\text{-}\mathrm{Ind}_{TG_{y,s}}^{G}\rho_G(\Psi)$$

is an irreducible supercuspidal representation (of depth r if χ is trivial, else of depth equal to that of χ).

We omit the subscript G where there is no possibility of confusion.

4.4. **Properties of the parametrization.** J. Hakim and F. Murnaghan [7] determined when two G-data give rise to equivalent supercuspidal representations, modulo a hypothesis called $C(\vec{G})$, which is satisfied in the toral case. We summarize their results for the particular G-data we consider here.

Proposition 4.3 (Hakim-Murnaghan)**.** *Let $\Psi = (T, y, \phi, r, \chi)$ and $\Psi' = (T', y', \phi', r', \chi')$ be two (toral, length-one, generic) G-data. Then*

(1) If $T = T'$, $r = r'$ and $\chi\phi = \chi'\phi'$, then $\rho(\Psi) \cong \rho(\Psi')$.
(2) We have $\pi(\Psi) \cong \pi(\Psi')$ if and only if there exists $g \in G$ such that $T' = {}^gT$, $r = r'$ and $\chi'\phi' = {}^g(\chi\phi)$ as characters of T'.

The first statement is an example of refactorization, and thus follows from [7, Proposition 4.24]. The second, incorporating G-conjugacy, is [7, Corollary 6.10]. The proofs of these results involve a detailed and complex analysis of the construction of ρ_G vis-à-vis defined notions of elementary transformations, refactorization and G-conjugacy.

We note the following additional properties of the construction.

Proposition 4.4. *Let $\Psi = (T, y, \phi, r, \chi)$ be a G-datum.*

(a) The restriction of $\rho(\Psi)$ to $Z \subset T$ is $\chi\phi$-isotypic.
(b) If $\Psi' = (T, y, \phi', r, \chi')$ is another G-datum such that $\mathrm{Res}_{T_r}\phi = \mathrm{Res}_{T_r}\phi'$ then the corresponding pullbacks of the Heisenberg-Weil representation are the same, that is, $\widetilde{\phi} = \widetilde{\phi}'$.
(c) If $\Psi' = (T, y, \phi', r, \chi')$ is another G-datum such that $\mathrm{Res}_{T_0}\phi = \mathrm{Res}_{T_0}\phi'$ then $\mathrm{Res}_{T_0 G_{y,s}}\widetilde{\rho}(\Psi) = \mathrm{Res}_{T_0 G_{y,s}}\widetilde{\rho}(\Psi')$.
(d) Set $S = G^1 \cap T$ and suppose that $\mathrm{Res}_{G^1}\chi = 1$. Then $\Psi^1 := (S, y, \mathrm{Res}_S\phi, r)$ is a generic toral length-one G^1-datum and $\mathrm{Res}_{SG^1_{y,s}}\rho_G(\Psi) = \rho_{G^1}(\Psi^1)$.

Proof. We adopt the notation of Section 4.2.

Part (a) is immediate from the construction if $TG_{y,s} = TG_{y,s+}$. Otherwise, since the conjugation action of Z on J and hence on W is trivial, $\tau_S \circ f$ is trivial on Z.

For part (b) we assume $TG_{y,s} \neq TG_{y,s+}$. Note that if ϕ and ϕ' are characters of depth r coinciding on T_r, then they induce the same character of T_r/T_{r+}. Thus part (b) is the observation that the dependence of $\widetilde{\phi}$ on ϕ in Section 4.2 is limited to the restriction of ϕ to this quotient.

Part (c) follows immediately from part (b), and the definition of $\widetilde{\rho}$.

To prove part (d), we show that the construction in Section 4.2 commutes with restriction to G^1, as follows.

First note that $\mathbb{S} = \mathbb{T} \cap \mathbb{G}^1$ is a minisotropic maximal torus of $\mathbb{G}^1$ which we may without loss of generality associate to the same point y of $\mathcal{B}^{red}(\mathbb{G}^1, F) = \mathcal{B}(\mathbb{G}^1, F) = \mathcal{B}^{red}(\mathbb{G}, F)$. Setting $S = \mathbb{S}(F)$, the character $\phi^1 := \mathrm{Res}_S\phi$ is also G^1-generic of depth r, by Proposition 2.2. Thus Ψ^1 is a (toral, generic, length-one) G^1-datum.

Let E be a splitting field of T and S, and denote the groups arising in the construction for G^1 with the superscript 1. Since for each root α, $\mathbb{G}_\alpha(E)_{y,s} = \mathbb{G}^1_\alpha(E)_{y,s}$, the groups $J^1(E)$ and $J^1_+(E)$ are defined as for $\mathbb{G}$ but with $\mathbb{T}(E)_r$ replaced by $\mathbb{S}(E)_r$. It follows that $J^1 \subseteq J$ and $\mathrm{Res}_{J^1_+}\widehat{\phi} = \widehat{\phi}^1$. Since $J^1 = J^1_+$ if and only if $J = J_+$, the result follows directly in this case.

So suppose $J \neq J_+$. Since $J_+ \cap J^1 = J^1_+$ and $J^1 J_+ = J$, the inclusion $\iota\colon J^1 \to J$ induces an isomorphism $\beta\colon W^1 \to W$. Moreover, the symplectic forms on these spaces induced by $\widehat{\phi}$ and $\widehat{\phi}^1$ coincide under β, since $\mathrm{Res}_{J^1_+}\widehat{\phi} = \widehat{\phi}^1$, so β is a symplectic isomorphism. Since $f = f_\nu$ is given by the conjugation action of T on W, and similarly for $f^1 = f^1_{\nu^1}$, it follows that $inn(\beta) \circ f^1 = f$ on S.

Next, since $\ker \widehat{\phi}^1 = \ker \widehat{\phi} \cap J^1_+$, we have $N^1 = N \cap J^1_+$. The genericity of ϕ further implies that $T_r/T_{r+} = S_r(N \cap T_r)/T_{r+}$, so the induced map $\alpha\colon H^1 \to H$ is again an isomorphism. Since ν_E and ν_E^1 are constructed from the same polarization, it follows that $\nu \circ \alpha = (\beta \times id) \circ \nu^1$.

Let (τ, V) be a Heisenberg representation of $W^\sharp$ with central character κ^{-1} and set $\tau' = \tau \circ (\beta \times id)$. It follows from Proposition 3.3 that the corresponding pullback representations of $S \ltimes J^1$ corresponding to τ and τ' are equal. Since $\tau' \cong \tau^1$, we deduce that $\mathrm{Res}_{S \ltimes J^1} \widetilde{\phi} \cong \widehat{\phi}^1$, and the result. □

We call the G^1-datum of part (d) the *restriction* of Ψ to G^1.

5. On restrictions of representations of G to G^1

Suppose that ZG^1 is open of finite index in G; if F is a p-adic field (of characteristic zero) this is immediate, by an argument using Galois cohomology. Then the restriction of any irreducible representation of G to G^1 decomposes as a finite direct sum of irreducible representations.

Let $\Psi = (T, y, \phi, r, \chi)$ and suppose that $\mathrm{Res}_{G^1}\chi$ is trivial. By Proposition 4.4(d), it follows that the irreducible representation $\pi_{G^1}(\Psi^1)$ occurs in $\mathrm{Res}_{G^1}\pi_G(\Psi)$, where Ψ^1 denotes the restriction of Ψ to G^1. Since G^1 is normal in G, the remaining summands each have the form ${}^\gamma\pi_{G^1}(\Psi^1)$, for some $\gamma \in G$. On the other hand by Proposition 4.3, $\pi_G(\Psi) \cong \pi_G({}^\gamma\Psi)$, so it follows that $\pi_{G^1}({}^\gamma\Psi^1)$ also occurs as a summand of $\mathrm{Res}_{G^1}\pi_G(\Psi)$.

Theorem 5.1. *Let Ψ^1 be the restriction to G^1 of a G-datum (T, y, ϕ, r, χ) such that* $\mathrm{Res}_{G^1}\chi = 1$. *Then for each $\gamma \in G$ we have*

$$ {}^\gamma\pi_{G^1}(\Psi^1) \cong \pi_{G^1}({}^\gamma\Psi^1). $$

Proof. Let $\Psi = (T, y, \phi, r, \chi)$ be a G-datum such that $\mathrm{Res}_{G^1}\chi = 1$. For any $\gamma \in G$, ${}^\gamma\Psi := ({}^\gamma T, \gamma \cdot y, {}^\gamma\phi, r, \chi)$ is again a G-datum.

Set $S = T \cap G^1$ and $\phi^1 = \mathrm{Res}_S\phi$; then $\Psi^1 = (S, y, \phi^1, r)$. As ${}^\gamma T \cap G^1 = {}^\gamma S$, we have $\mathrm{Res}_{{}^\gamma S}{}^\gamma\phi = {}^\gamma\phi^1$. Therefore the restriction of ${}^\gamma\Psi$ to G^1 coincides with the twisted datum ${}^\gamma\Psi^1 = ({}^\gamma S, \gamma \cdot y, {}^\gamma\phi^1, r)$.

Note that

$$ {}^\gamma\pi(\Psi^1) = {}^\gamma\left(\text{c-Ind}_{SG^1_{y,s}}^{G^1}\rho(\Psi^1)\right) \cong \text{c-Ind}_{({}^\gamma S)G^1_{\gamma\cdot y,s}}^{G^1}{}^\gamma\rho(\Psi^1) $$

so it suffices to show that ${}^\gamma\rho_{G^1}(\Psi^1) \cong \rho_{G^1}({}^\gamma\Psi^1)$ as representations of ${}^\gamma SG^1_{\gamma\cdot y,s}$. In fact it is enough to show that if $\widetilde{\phi}$ and $\widetilde{\phi}^1$ are the pullbacks of corresponding Heisenberg-Weil representations then ${}^\gamma\widetilde{\phi} \cong \widetilde{\phi}^1$ on ${}^\gamma S \ltimes {}^\gamma J^1$.

Let J^1 and J^1_+ denote the subgroups of G^1 corresponding to Ψ^1; it follows from their construction that ${}^\gamma J^1$ and ${}^\gamma J^1_+$ are the subgroups of G^1 corresponding to ${}^\gamma\Psi^1$. Thus we are done in the case that $SJ^1 = SJ^1_+$.

So suppose $SJ^1 \neq SJ^1_+$, and consider the construction in Section 4.2. Use a subscript γ to denote an object in the construction corresponding to the datum ${}^\gamma\Psi^1$. The character $\widehat{(\phi_E)_\gamma}$ of ${}^\gamma J^1_+(E)$ coincides with ${}^\gamma\widehat{\phi_E}$, whose kernel is $N(E)_\gamma = {}^\gamma N(E)$. Similarly, we have $H(E)_\gamma = {}^\gamma H(E)$ and $W(E)_\gamma = {}^\gamma W(E)$. Moreover, the symplectic form on $W(E)_\gamma$ is given by $\langle x, y\rangle_\gamma = \langle {}^{\gamma^{-1}}x, {}^{\gamma^{-1}}y\rangle$; it follows that the polarization used in the construction of ν_γ is $W(\pm)_\gamma \cong {}^\gamma W(\pm)$.

Thus conjugation by γ defines isomorphisms $\alpha\colon H(E) \to H(E)_\gamma$, $\beta\colon W(E) \to W(E)_\gamma$ and $\delta\colon \mathbb{S}(E) \to {}^\gamma\mathbb{S}(E)$. We show that the hypotheses of Proposition 3.3 hold.

Let $h \in H(E)$ and factor h as $g_+g_-g_0$ following the polarization; then $\alpha(h)$ factors as ${}^\gamma g_+ {}^\gamma g_- {}^\gamma g_0$. Since $\kappa(\widehat{\phi_E}(g_0)) = \kappa({}^\gamma\widehat{\phi_E}({}^\gamma g_0))$ and $\langle g_+, g_-\rangle = \langle {}^\gamma g_+, {}^\gamma g_-\rangle_\gamma$, we have $\mu(h) = \mu({}^\gamma h)$. Descending now to F, we deduce that for all $h \in H$,

$$(\beta \times id)\nu(h) = (\beta(hZ_H), \mu(h)) = ({}^\gamma hZ_{H_\gamma}, \mu({}^\gamma h)) = \nu_\gamma(\alpha(h)).$$

Next, let ${}^\gamma w \in {}^\gamma W$ and $t \in S$. Then

$$\beta \circ f(t) \circ \beta^{-1}({}^\gamma w) = \beta(twt^{-1}) = f({}^\gamma t)({}^\gamma w)$$

so that $inn(\beta) \circ f = f \circ \delta$.

Let (τ_γ, V) be a Heisenberg representation of $W^\sharp_\gamma$ with central character κ^{-1}, and set $\tau' = \tau_\gamma \circ (\beta \times id)$. Denoting by $\widetilde{\phi}'$ the pullback of the Heisenberg-Weil representation corresponding to τ', we conclude from Proposition 3.3 that for all $(s, j) \in S \ltimes J^1$,

$$\widetilde{\phi}_\gamma({}^\gamma s, {}^\gamma j) = \widetilde{\phi}'(s, j)$$

and thus that ${}^\gamma\widetilde{\phi}' = \widetilde{\phi}_\gamma$ on ${}^\gamma S \ltimes {}^\gamma J^1$. Replacing τ' with the equivalent representation τ yields ${}^\gamma\widetilde{\phi} \cong \widetilde{\phi}_\gamma$, whence the result. □

Corollary 5.2. *Let $\Psi = (T, y, \phi, r, \chi)$ be a G-datum such that $\mathrm{Res}_{G^1}\chi$ is trivial and let Ψ^1 denote its restriction to G^1. Then*

$$\mathrm{Res}_{G^1}\pi(\Psi) \cong \bigoplus_{\gamma \in G/TG^1} \pi({}^\gamma\Psi^1).$$

Proof. By Theorem 5.1 and the remarks preceding it, we may apply Mackey theory to deduce that

$$\begin{aligned}\mathrm{Res}_{G^1}\pi_G(\Psi) &= \mathrm{Res}_{G^1}\text{c-Ind}_{TG_{y,s}}^G \rho_G(\Psi) \\ &\cong \bigoplus_{\gamma \in G^1\backslash G/TG_{y,s}} \text{c-Ind}_{G^1 \cap {}^\gamma(TG_{y,s})}^{G^1} {}^\gamma\rho_G(\Psi) \\ &\cong \bigoplus_{\gamma \in G^1\backslash G/TG_{y,s}} \pi_{G^1}({}^\gamma\Psi^1).\end{aligned}$$

As G^1 is normal in G, and $TG_{y,s} = TG^1_{y,s}$, the given decomposition follows. Suppose that for some $\gamma \in G \setminus TG^1$ we had ${}^\gamma S = S$, ${}^\gamma\phi^1 = \phi^1$ and $\gamma \cdot y = y$ in $\mathcal{B}^{red}(\mathbb{G}, F)$. Then as ${}^\gamma\phi^1$ and ϕ^1 extend to the same character of $S_{0+}G^1_{y,s+}$, and $T_{0+} = Z_{0+}S_{0+}$ [6, Lemma B.7.2], we conclude that on the subgroup $T_{0+}G_{y,s+}$, we have ${}^\gamma\phi = \phi$. Thus by [14, Prop 4.1], $\gamma \in (T_0G_{y,s})T(T_0G_{y,s}) \subseteq TG^1$. It follows now from Proposition 4.3 that the summands are distinct. □

6. Application to the multiplicative group of the quaternion algebra over F

Let D be the quaternionic division algebra over F. We recall the groups $D^\times$ and D^1 and cast their (well-known) representation theory in the language of the preceding sections. We assume $p > 2$; this satisfies all the hypotheses in Section 2. In this section we prove only results which are not readily found in the literature.

6.1. **Notation and background on $D^\times$.** Let ε denote a nonsquare in $\mathcal{O}_F^\times$. Then the quaternion algebra $D = \mathbb{D}(F)$ over F can be realized as the F-algebra with presentation

$$\langle 1, i, j, k \mid i^2 = \varepsilon, j^2 = \varpi, k^2 = -\varepsilon\varpi, ij = k = -ji\rangle.$$

Given $z = a + bi + cj + dk$ in this presentation, the anti-involution $z \mapsto \overline{z} = a - bi - cj - dk$ defines the (reduced) trace as $\mathrm{Tr}(z) = 2a$ and the (reduced) norm as $\mathrm{nrd}(z) = a^2 - b^2\varepsilon - c^2\varpi + d^2\varepsilon\varpi$, both taking values in F. The ring $\mathcal{O}_D = \{z \in D \mid \mathrm{nrd}(z) \in \mathcal{O}_F\}$ is a maximal compact open subring with unique maximal ideal $\mathcal{P}_D = \mathcal{O}_D j$. We normalize our valuation in F so that $\mathrm{val}(\varpi) = 1$ and extend it to a valuation, also denoted val, on D, or any algebraic extension field of F. In particular note that $\mathrm{val}(j) = \frac{1}{2}$.

The map nrd is algebraic, and the derived group of $\mathbb{D}^\times$ is $\mathbb{D}^1 = \ker(\mathrm{nrd})$. The groups $D^\times = \mathbb{D}^\times(F)$ and $D^1 = \mathbb{D}^1(F) \subseteq \mathcal{O}_D{}^\times$ are both residually quasi-split and compact mod centre. The Lie algebra of $D^\times$ is D whereas that of D^1 consists of elements of trace zero. One has $[D^\times, D^\times] = D^1$ [9, Lemma I.4.1] and $[D^1, D^1] = D^1 \cap (1 + \mathcal{P}_D)$ [13, §5]. The center of $D^\times$ is $Z = F^\times$ and thus ZD^1 has index equal to $|F^\times/F^{\times 2}| = 4$ in $D^\times$.

Each quadratic extension E of F can be embedded in D, uniquely up to $D^\times$-conjugacy, and the restriction of the anti-involution $\bar{\cdot}$ to E coincides with the action of the nontrivial Galois element. Furthermore, for each such E there is some $\sigma \in D^\times$ such that ${}^\sigma z = \overline{z}$ for all $z \in E$. Note that $E^1 := E^\times \cap D^1$ is given by $\{\beta\overline{\beta}^{-1} \mid \beta \in E\}$.

One may choose explicit representatives as follows. Denote by L the unramified extension field $F[i]$ contained in D; then one may take $\sigma = j$. Its residue field is denoted $\mathfrak{l}$, whose norm one elements form the group $\mathfrak{l}^1$. Fix $\mu \in L^\times$ satisfying $\mathrm{nrd}(\mu) = \varepsilon$; if $-1 = z^2$ for some $z \in F$ then we may choose $\mu = zi$. It follows that the two nonconjugate ramified extensions of F in D are represented by $F[j]$ and $F[\mu j]$; in these cases one may take $\sigma = i$.

The maximal tori of $D^\times$ are exactly the groups $E^\times$, for E a quadratic extension of F; there are thus three conjugacy classes. For each maximal torus T of $D^\times$, it follows that TD^1 has index 2 in $D^\times$ (by the nrd map) and that $N_{D^\times}(T) = T \sqcup T\sigma$.

Lemma 6.1. *There are three conjugacy classes of maximal tori of D^1 when $-1 \in (F^\times)^2$. Otherwise, for each ramified torus T of $D^\times$, the tori $D^1 \cap T$ and $D^1 \cap {}^\mu T$ are not D^1-conjugate, and there are a total of five D^1-conjugacy classes.*

Proof. For each maximal torus S of D^1 there is a maximal torus T of $D^\times$ such that $S = T \cap D^1$. For fixed T, the set of D^1-conjugacy classes of tori in $\{{}^\gamma T \cap D^1 \mid \gamma \in D^\times\}$ is parametrized by $\gamma \in D^\times / N_{D^\times}(T)D^1$. This group is nontrivial if and only if T is ramified and $-1 \notin {F^\times}^2$, in which case it has order two and a set of representatives is $\{1, \mu\}$. □

From this one deduces that all maximal tori in D^1 are self-normalizing.

6.2. **Genericity of quasi-characters of tori.** The homomorphism $\varphi\colon D^\times \to \mathrm{GL}_2(L)$ determined by $\varphi(i) = \left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right]$, $\varphi(j) = \left[\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right]$ is an embedding. Its image in $\mathrm{GL}_2(L)$ is the set of fixed points under the involution $\Theta(g) = \varphi(j)^{-1}\overline{g}\varphi(j)$. Thus we can realize the reduced building $\mathcal{B}^{red}(\mathbb{D}, F)$ of $D^\times$ as the unique fixed point x in $\mathcal{B}^{red}(\mathrm{GL}_2, L)$ of the automorphism Θ. For this choice of φ, the diagonal split torus is Θ-stable, and x lies in the corresponding apartment $\mathcal{A} \subset \mathcal{B}^{red}(\mathrm{GL}_2, L)$, where it is the barycentre of the fundamental chamber. We can and do omit the subscript x from our notation in this case. For $G \in \{D^\times, D^1\}$ the Moy-Prasad filtration subgroups are simply given by $G_r = \{g \in G \mid \mathrm{val}_D(g) \geq r\}$ and $G_{r+} = \{g \in G \mid \mathrm{val}_D(g) > r\}$. Note that $D_0^\times = \mathcal{O}_D^\times$ and $D_0^1 = D^1$.

The notion of a generic character of a maximal torus of $D^\times$ coincides with the original notion of an admissible character, due to R. Howe [8], as follows.

Lemma 6.2. *Any nontrivial quasi-character of a maximal torus of D^1 is D^1-generic. For T a maximal torus of $D^\times$, the quasi-character ϕ of T of positive depth r is $D^\times$-generic if and only if $r = \min\{depth(\chi\phi)\colon \chi \in \widehat{F^\times}\}$ where $\chi\phi := (\chi \circ \mathrm{nrd}) \otimes \phi$.*

Proof. Let ϕ be a quasi-character of T of depth $r > 0$. As $\mathfrak{s}$ is one-dimensional over F, a character of S is D^1-generic if and only if it is nontrivial. Therefore by Proposition 2.2, ϕ is $D^\times$-generic of depth r if and only if $\mathrm{Res}_{S_r}\phi$ is nontrivial. Thus in particular if for any $\chi \in \widehat{F^\times}$ the depth of $\chi\phi$ were less than r, then $\mathrm{Res}_{S_r}\phi = \mathrm{Res}_{S_r}\chi\phi = 1$, whence ϕ would not be G-generic of depth r.

Conversely, suppose $\mathrm{Res}_{S_r}\phi = 1$, and suppose $X^* \in \mathfrak{t}^*_{-r}$ realizes ϕ. Decompose $X \in \mathfrak{t}_r$ as $Z + Y$ with $Y \in \mathfrak{s}_r$ and $Z \in \mathfrak{z}_r$; then as $\phi(e(Y)) = 1$ we have $\phi(e(X)) = \phi(e(Z)) = \Psi(\langle X^*, Z\rangle)$. Note that $\mathrm{nrd}(e(X)) = \mathrm{nrd}(e(Z))$. As $\mathfrak{z} = F \subset D$, we may identify Z with an element $z \in \mathcal{P}_F^r$, which under e is mapped to the class of $1 + z \in Z_r \subset T_r$ modulo T_{r+}. Then since $r > 0$, we have $\mathrm{nrd}(e(Z)) \equiv 1 + 2z$ modulo $\mathcal{P}_F^{r+}$. Choose a character χ of $F^\times$ of depth r satisfying $\chi(1+b) = \Psi(-\langle X^*, \frac{1}{2}b\rangle)$ for all

$b \in \mathcal{P}_F^r \subset \mathfrak{z}$. Then $\chi\phi(e(X)) = \chi(\mathrm{nrd}(e(X))\phi(e(X)) = \Psi(-\langle X^*, \frac{1}{2}2z\rangle)\phi(e(Z)) = 1$, whence $\chi\phi$ is trivial on T_r, and so has depth strictly less than r. □

6.3. **Depths of generic quasi-characters of tori.** Let $G \in \{D^\times, D^1\}$.

Proposition 6.3. *Let T be a maximal torus of G. If T is unramified then its G-generic characters have integral depth, whereas if T is ramified then its G-generic characters have depth in $\frac{1}{2} + \mathbb{Z}$.*

Proof. Each maximal torus T of $D^\times$ has the form $F[\beta]^\times \subset D^\times$, for some $\beta \in D \setminus F$; we may without loss of generality assume β has trace 0. Then $\mathrm{val}(\beta) \in \mathbb{Z}$ if T is unramified and $\mathrm{val}(\beta) \in \frac{1}{2} + \mathbb{Z}$ otherwise. The Lie algebra of $S = T \cap D^1$ is $\mathfrak{s} = F\beta$. Thus the values $r \in \mathbb{R}$ for which $\mathfrak{s}_r \neq \mathfrak{s}_{r+}$ are integral if T is unramified, and lie in $\frac{1}{2} + \mathbb{Z}$ if T is ramified. The result for S and T now follows from Lemma 6.2 and Proposition 2.2, respectively. □

Corollary 6.4. *Let T be a maximal torus of G and ϕ a G-generic quasi-character of T of depth r. Set $s = r/2$. Then $TG_s = TG_{s+}$ unless T is unramified and r is odd.*

Proof. We note that $G_s = G_{s+}$ unless $s \in \frac{1}{2}\mathbb{Z}$, and therefore by Proposition 6.3, the equality follows for T a ramified torus. If $G = D^\times$ and T is unramified then we can decompose $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}\gamma$, where $\mathrm{val}(\gamma) = \frac{1}{2}$, whence upon passing through the Lie algebra we deduce that for each integral s, $G_s/G_{s+} \cong T_s/T_{s+}$. Thus $TG_s = TG_{s+}$ whenever $s \in \mathbb{Z}$. The same argument holds for $G = D^1$ and $S = T \cap D^1$, by noting the analogous decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{t}j$. Finally, since $T \cap D_s^\times = T_{\lceil s \rceil}$ it follows that $TD_s^\times \neq TD_{s+}^\times$, and hence that $SD_s^1 \neq SD_{s+}^1$, for $s \in \frac{1}{2} + \mathbb{Z}$. □

6.4. **Smooth representations of $D^\times$ and D^1.** The smooth irreducible representations of $D^\times$ and D^1 are well-known, see [4, 8, 3] and [11] respectively. We present the complete list for the case of $p \neq 2$ (the *tame* case) here; the case $p = 2$ for includes more representations and for $D^\times$ is treated in, for example, [3, Ch 13]. For simplicity, we reserve "representation of depth $\star$" for the subset of those of degree greater than one (that is, excluding the quasi-characters).

6.4.1. *Characters.* Since $D^\times/[D^\times, D^\times] \cong D^\times/D^1 \cong F^\times$ the one-dimensional smooth representations of $D^\times$ are in bijection with characters of $F^\times$ via the nrd map. On the other hand, as $[D^1, D^1] = D^1 \cap (1 + \mathcal{P}_D) = D_{0+}^1$, and $D^1/D_{0+}^1 \cong \mathfrak{l}^1$, there are $q + 1$ distinct characters of D^1, each of depth 0.

6.4.2. *Depth-zero representations.* Since $D^1 = D_0^1$, the depth-zero representations of D^1 are those which factor through D^1/D_{0+}^1, namely its characters, so by our convention we will say D^1 has no representations of depth zero.

In contrast $D^\times$ admits depth-zero representations, whose construction we summarize from [3, §54.2] as follows. A depth-zero generic or *admissible* character of $L^\times$ (see

[8]) is a quasi-character θ of $L^\times$ of depth zero which does not factor through the norm map, or equivalently, such that $\theta \neq \overline{\theta}$ where $\overline{\theta}(z) = \theta(\overline{z})$. Two admissible characters θ' and θ are called F-equivalent if $\theta' = \overline{\theta}$.

Given a depth-zero admissible character θ of $T = L^\times$, extend it trivially across D_{0+} to give a quasi-character θ of $TD_{0+}^\times$. Then

$$\pi(\theta) = \mathrm{Ind}_{TD_{0+}^\times}^{D^\times} \theta$$

is an irreducible representation of $D^\times$ of depth zero. Moreover, isomorphism classes of depth-zero representations of $D^\times$ are in bijection with F-equivalence classes of depth-zero admissible characters of $L^\times$.

Remark 6.5. These representations are very simple. In fact $TD_{0+}^\times = L^\times(1 + \mathcal{P}_D) = L^\times D^1$, which has one nontrivial coset in $D^\times$, represented by σ. Thus π has degree 2 and is just the corresponding extension to $D^\times$ of $\theta \oplus \overline{\theta}$.

6.4.3. *Positive-depth representations.* Let $G \in \{D^\times, D^1\}$. There are two kinds of representations of positive depth of G. The first is immediate: given any supercuspidal representation π_0 of depth zero of G, and any character χ of positive depth of G, set $\pi = \chi\pi_0$.

The second kind are parametrized by generic toral G-data Ψ of length one, as in Section 4. If $G = D^\times$ we identify the central character χ with a character of $F^\times$ whereas if $G = D^1$, then following the definition we have $\chi = 1$, which we may omit. In both cases y is forced to be the unique point x in $\mathcal{B}^{red}(G, F)$, so we may omit it as well.

7. Restrictions of representations of $D^\times$

In this section we apply the results of Section 5 to determine the restrictions and decomposition into irreducible representations of each of the representations of $D^\times$ to ${\mathcal{O}_D}^\times$ and to D^1. The restriction to D^1 has presumably been known to experts; that to $\mathcal{O}_D^\times$ allows the classification of its irreducible representations in Section 8.

7.1. Branching rules for the restriction of representations of $D^\times$ to D^1.

The restriction of any character of $D^\times$ to D^1 is trivial, since $D^1 = \mathbb{D}^1(F)$.

Lemma 7.1. *All nontrivial characters of D^1 occur in the restriction of a depth-zero representation of $D^\times$.*

Proof. Let θ be a depth-zero admissible character of $T = L^\times$ and set $\vartheta = \mathrm{Res}_{D^1 \cap L^\times} \theta$, which may be identified with a character of $\mathfrak{l}^1$. By Remark 6.5 we conclude that $\mathrm{Res}_{D^1}\pi(\theta)$ decomposes as $\vartheta \oplus \overline{\vartheta}$. As $L^1 = \{z\overline{z}^{-1} \mid z \in L^\times\}$, the admissibility of θ is equivalent to $\vartheta \neq 1$. □

The decomposition of positive-depth representations of $D^\times$ upon restriction to D^1 is determined by. For a D^1-datum $\Psi^1 = (S, \phi^1, r)$ we write $\overline{\Psi^1} = (S, (\phi^1)^{-1}, r)$. The ubiquitious hypothesis $\mathrm{Res}_{G^1}\chi = 1$ on the datum Ψ is always satisfied in this case.

Proposition 7.2. *Let Ψ be a $D^\times$ datum and let Ψ^1 denote its restriction to D^1. Then $\mathrm{Res}_{D^1}\pi(\Psi)$ decomposes as a direct sum of two inequivalent representations. When Ψ is unramified, or when $-1 \in F^{\times 2}$, we have*

$$\mathrm{Res}_{D^1}\pi(\Psi) \cong \pi(\Psi^1) \oplus \pi(\overline{\Psi^1})$$

whereas otherwise, writing $\Psi^1 = (S, \phi^1, r)$, we have

$$\mathrm{Res}_{D^1}\pi(\Psi) \cong \pi(S, \phi^1, r) \oplus \pi({}^{\mu}S, {}^{\mu}\phi^1, r).$$

Proof. It suffices by Corollary 5.2 to note that $D^\times/TD^1$ is represented by $\{1, \sigma\}$ except in the case that T is ramified and $-1 \notin F^{\times 2}$, where it is represented by $\{1, \mu\}$. □

7.2. **Branching rules for the restriction of representations of $D^\times$ to $\mathcal{O}_D^\times$.** Note that the center of $\mathcal{O}_D^\times$ is $\mathcal{O}_F^\times$. Therefore since $\mathcal{O}_F^\times D^1$ has index two in $\mathcal{O}_D^\times$, and $F^\times\mathcal{O}_D^\times$ has index two in $D^\times$, each restriction, from $D^\times$ to $\mathcal{O}_D^\times$, or from $\mathcal{O}_D^\times$ to D^1, is either irreducible or else a direct sum of two inequivalent irreducible representations. We may thus deduce many of the branching rules for $\mathcal{O}_D^\times$ from the results of the preceding section.

We begin with the characters.

Lemma 7.3. *Each character of $\mathcal{O}_D^\times$ may be uniquely factored as $\chi\theta := (\chi \circ \mathrm{nrd})\theta$, where $\chi \in \widehat{\mathcal{O}_F^\times}$ and θ either trivial, or else the inflation of an admissible depth zero character of $\mathcal{O}_L^\times$ to $\mathcal{O}_D^\times$.*

Proof. Restricting a character χ of $D^\times$ to $\mathcal{O}_D^\times$ gives $\chi_0\circ\mathrm{nrd}$, where $\chi_0 = \mathrm{Res}_{\mathcal{O}_F^\times}\chi \in \widehat{\mathcal{O}_F^\times}$. Next, let θ be an admissible character of the unramified torus $L^\times$. Since $\mathcal{O}_D^\times/(1+\mathcal{P}_D) \cong L_0^\times/L_{0+}^\times$, we may view θ as a depth-zero character of $\mathcal{O}_D^\times$. By Remark 6.5, it follows that $\mathrm{Res}_{\mathcal{O}_D^\times}\pi(\theta) = \theta \oplus \overline{\theta}$. Finally, restricting the positive-depth representation (of the first kind) $\chi\pi(\theta)$ to $\mathcal{O}_D^\times$ yields $\chi_0\theta \oplus \chi_0\overline{\theta}$.

Each character of $\mathcal{O}_D^\times$ must occur in the restriction of a representation of $D^\times$. Since $D^1_{0+} = [D^1, D^1] \subseteq [\mathcal{O}_D^\times, \mathcal{O}_D^\times]$ we deduce our list of characters is exhaustive; the unicity of the factorization follows since admissible characters are exactly those which do not factor through the norm map. □

We now turn to the restrictions of representations of positive depth of $D^\times$ (of the second kind).

Proposition 7.4. *Let $\Psi = (T, \phi, r, \chi)$ be an unramified $D^\times$-datum. Then*

$$\mathrm{Res}_{\mathcal{O}_D^\times}\pi(\Psi) \cong \mathrm{Ind}_{T_0D_s^\times}^{\mathcal{O}_D^\times}\rho(\Psi) \oplus \mathrm{Ind}_{T_0D_s^\times}^{\mathcal{O}_D^\times}\rho(\overline{\Psi})$$

is a decomposition into irreducible inequivalent representations of $\mathcal{O}_D^\times$.

Proof. Let $\pi = \pi(\Psi)$ and $\rho = \rho(\Psi)$. We have $\mathcal{O}_D{}^\times \backslash D^\times / TD_s^\times = \{1, \sigma\}$ so by Mackey theory and the remarks above a decomposition of $\mathrm{Res}_{\mathcal{O}_D^\times}\pi$ into irreducible representations is

$$\mathrm{Res}_{\mathcal{O}_D{}^\times}\pi \cong \mathrm{Ind}_{T_0 D_s^\times}^{\mathcal{O}_D^\times}\rho \oplus \mathrm{Ind}_{T_0 D_s^\times}^{\mathcal{O}_D^\times}{}^\sigma\rho$$

where we have used that $\mathcal{O}_D{}^\times \cap TG_{y,s} = T_0 D_s^\times$ and that σ normalizes this group. The inequivalence of the factors follows, for example, from that of their further restriction to D^1, as per Proposition 7.2. $\square$

Note that it follows from Proposition 7.2 that for an unramified $D^\times$-datum Ψ with restriction Ψ^1 to D^1, we have

$$\mathrm{Res}_{\mathcal{O}_D^\times}\mathrm{Ind}_{T_0 D_s^\times}^{\mathcal{O}_D^\times}\rho(\Psi) \cong \pi_{D^1}(\Psi^1).$$

Proposition 7.5. *Let $\Psi = (T, \phi, r, \chi)$ be a ramified $D^\times$-datum, $s = r/2$ and $\pi = \pi(\Psi)$. Then*

$$\mathrm{Res}_{\mathcal{O}_D^\times}\pi \cong \mathrm{Ind}_{T_0 D_s^\times}^{\mathcal{O}_D{}^t}\rho(\Psi) \tag{7.1}$$

is irreducible.

Proof. Write $G = D^\times$ and let $T = E^\times$, where $E = F[\beta]$ for some $\beta \in D$ with $\mathrm{val}_D(\beta) = \frac{1}{2}$. Then $\mathfrak{t} = E$, with T-invariant complement Ei in $\mathfrak{g}$. Since $i \in \mathcal{O}_D^\times$, we deduce $D^\times = \mathcal{O}_D^\times(TG_s)$; also note that $\mathcal{O}_D^\times \cap TG_s = T_0 G_s$. This yields the isomorphism (7.1).

To show this representation is irreducible, we show for each $\gamma \in \mathcal{O}_D^\times$ that the space of intertwining operators

$$\mathrm{Hom}_{T_0 G_s \cap {}^\gamma(T_0 G_s)}(\rho, {}^\gamma\rho)$$

is zero unless $\gamma \in T_0 G_s$.

The representation $\rho = \rho(\Psi)$ is the character $\chi\widehat{\phi}$, where $\widehat{\phi}$ is the extension of ϕ to TG_s which is trivial on the T-invariant complement of $T \cap G_s$ in G_s. More explicitly, decompose $g \in G_s$ as $g = 1 + b + ci$ with $b, c \in E_s$; then for any $a \in T$, we have

$$\widehat{\phi}(ag) = \widehat{\phi}(a(1 + b + ci)) = \phi(a(1 + b)).$$

Now let $\gamma \in \mathcal{O}_D^\times \setminus T_0 G_s$. Decompose $\gamma = u + vi$ with $u, v \in \mathcal{O}_E$; then at least one of u or v is invertible and $\nu := \mathrm{val}_D(v) < s$. Set $n = \mathrm{nrd}(\gamma) \in \mathcal{O}_F^\times$ and let $a \in \mathcal{O}_E^\times$. We compute directly that $h = \gamma a \gamma^{-1} = a(1 + a^{-1}n^{-1}(a - \overline{a})v(\overline{v}\varepsilon - ui))$, which lies in $T_0 G_s$ whenever, for example,

$$\mathrm{val}_D((a - \overline{a})) \geq s - \nu. \tag{7.2}$$

Set $r_a = r - 2\nu \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. Then as in particular $r_a \geq s - \nu$ we deduce that ${}^\gamma T_{r_a} \subset (TG_s) \cap {}^\gamma(TG_s)$. Fixing such an $h = \gamma a \gamma^{-1} \in {}^\gamma T_{r_a}$, we have ${}^\gamma\rho(h) = \rho(a) = \chi(\mathrm{nrd}(a))\phi(a)$ whereas by construction

$$\begin{aligned}\rho(h) &= \chi(\mathrm{nrd}(h))\phi(a(1 + a^{-1}n^{-1}(a - \overline{a})v(\overline{v}\varepsilon - ui))) \\ &= \chi(\mathrm{nrd}(a))\phi(a)\phi(1 + b_a)\end{aligned}$$

where $b_a := a^{-1}n^{-1}(a-\overline{a})v\overline{v}\varepsilon$ lies in E_r. Since the map $a \mapsto 1+b_a$ induces a surjective map from T_{r_a} onto T_r/T_{r+}, and ϕ is nontrivial on T_r, we deduce that ρ and ${}^\gamma\rho$ are not equal on $(TG_s) \cap {}^\gamma(TG_s)$, as required. □

8. Classification of irreducible representations of $\mathcal{O}_D^\times$

Suppose $\Psi = (T, \phi, r, \chi)$ and $\Psi' = (T', \phi', r', \chi')$ are two $D^\times$-data. We say that Ψ and Ψ' are $\mathcal{O}_D^\times$-equivalent, and write $\Psi \equiv_{\mathcal{O}_D^\times} \Psi'$, if there exists $g \in \mathcal{O}_D^\times$ for which ${}^gT_0 = T'_0$, $r = r'$ and $\mathrm{Res}_{T'_0}{}^g(\chi\phi) = \mathrm{Res}_{T'_0}\chi'\phi'$. Note that since in this case $T_0 \cong \mathcal{O}_E^\times$ where $T = E^\times$, the condition ${}^gT_0 = T'_0$ is equivalent to ${}^gT = T'$.

Given Ψ as above we let $s = r/2$ and define $\pi_{\mathcal{O}_D^\times}(\Psi) = \mathrm{Ind}_{T_0G_s}^{\mathcal{O}_D^\times}\rho(\Psi)$, which is irreducible by Propositions 7.4 and 7.5.

Theorem 8.1. *The irreducible representations of ${\mathcal{O}_D}^\times$ are:*

(1) the characters: $\chi\theta := (\chi \circ \mathrm{nrd})\theta$, where $\chi \in \widehat{\mathcal{O}_F^\times}$ and θ is either trivial or the inflation to ${\mathcal{O}_D}^\times$ of an admissible depth-zero character of $L^\times \subset {\mathcal{O}_D}^\times$;
(2) the representations of degree greater than one: $\pi_{\mathcal{O}_D^\times}(\Psi)$, for a $D^\times$-datum $\Psi = (T, \phi, r, \chi)$.

Moreover,$\pi_{\mathcal{O}_D^\times}(\Psi) \cong \pi_{\mathcal{O}_D^\times}(\Psi')$ if and only if Ψ and Ψ' are $\mathcal{O}_D^\times$-equivalent.

Proof. The first point is Lemma 7.3. That the list in the second point is exhaustive follows from the classification of representations of $D^\times$ and Propositions 7.4 and 7.5. The equivalence among characters was established in Lemma 7.3, so we have only to prove the last statement.

Thus suppose $\Psi \equiv_{\mathcal{O}_D^\times} \Psi'$; let us show that $\pi_{\mathcal{O}_D^\times}(\Psi) \cong \pi_{\mathcal{O}_D^\times}(\Psi')$.

Since $\rho({}^g\Psi) = {}^g\rho(\Psi)$ for each $g \in D^\times$, it follows easily that for any $g \in \mathcal{O}_D^\times$, $\pi_{\mathcal{O}_D^\times}({}^g\Psi) \cong \pi_{\mathcal{O}_D^\times}(\Psi)$. Therefore WLOG we may replace Ψ' with an $\mathcal{O}_D^\times$-conjugate and thereby assume $T = T'$, $r = r'$ and $\mathrm{Res}_{T_0}\chi\phi = \mathrm{Res}_{T_0}\chi'\phi'$.

Set $\varphi = \chi'\phi'(\chi\phi)^{-1}$; since this is a character of T trivial on T_0, we deduce that $\Psi'' := (T, \varphi\phi, r, \chi)$ is also a generic $D^\times$-datum. Since $\chi\varphi\phi = \chi'\phi'$, Proposition 4.3 implies $\rho(\Psi'') \cong \rho(\Psi')$, whence their restrictions to T_0G_s are equivalent. On the other hand, since $\mathrm{Res}_{T_0}\varphi\phi = \mathrm{Res}_{T_0}\phi$, and the $D^\times$-characters of Ψ'' and Ψ coincide, it follows from Proposition 4.4 that $\mathrm{Res}_{T_0G_s}\rho(\Psi) = \mathrm{Res}_{T_0G_s}\rho(\Psi'')$. Consequently $\pi_{\mathcal{O}_D^\times}(\Psi) \cong \pi_{\mathcal{O}_D^\times}(\Psi')$.

Now suppose $\pi_{\mathcal{O}_D^\times}(\Psi) \cong \pi_{\mathcal{O}_D^\times}(\Psi')$. We need to show that $\Psi \equiv_{\mathcal{O}_D^\times} \Psi'$.

Set $S = T \cap D^1$ and $S' = T' \cap D^1$. Suppose $\pi_{\mathcal{O}_D^\times}(\Psi) \cong \pi_{\mathcal{O}_D^\times}(\Psi')$. Then as their restrictions to D^1 are isomorphic, it follows that T and T' are either both ramified, or both unramified. In the former case we have $\pi_{D^1}(\Psi^1) \in \{\pi_{D^1}(\Psi'^1), \pi_{D^1}({}^\mu\Psi'^1)\}$ and in the latter we have $\pi_{D^1}(\Psi^1) \cong \pi_{D^1}(\Psi'^1)$. In either case we may deduce that upon replacing Ψ with an $\mathcal{O}_D^\times$-conjugate that $T = T'$. Furthermore, since $\mu \in \mathcal{O}_D^\times$, we may replace Ψ' with ${}^\mu\Psi'$ if necessary and thus assume $\pi_{D^1}(\Psi^1) \cong \pi_{D^1}(\Psi'^1)$. Then it follows from Proposition 4.3 that $\mathrm{Res}_S\phi = \mathrm{Res}_S\phi'$.

On the other hand, as these representations have the same central character, we deduce that $\mathrm{Res}_{Z_0}\chi\phi = \mathrm{Res}_{Z_0}\chi'\phi'$. Thus $\chi\phi$ and $\chi'\phi'$ agree on Z_0S.

If T is ramified, then since for each $t \in T_0$, $\mathrm{nrd}(t) \in \mathcal{O}_F^{\times 2}$, we have $Z_0S = T_0$ and thus $\Psi \equiv_{\mathcal{O}_D^\times} \Psi'$.

If T is unramified, then Z_0S is of index two in T_0. Choose a character ξ of T which restricts on T_0 to the nontrivial character of T_0/Z_0S. Then it follows that $\mathrm{Res}_{T_0}\chi'\phi' \in \{\mathrm{Res}_{T_0}\chi\phi, \mathrm{Res}_{T_0}\xi\chi\phi\}$. We show that the second case cannot occur, whence one concludes $\Psi \equiv_{\mathcal{O}_D^\times} \Psi'$.

Namely, set $\Psi'' = (T, \xi\phi, r, \chi)$. Then $\Psi'' \equiv_{\mathcal{O}_D^\times} \Psi'$, so to derive a contradiction it suffices to show that $\pi_{\mathcal{O}_D^\times}(\Psi'') \not\cong \pi_{\mathcal{O}_D^\times}(\Psi)$. We may also WLOG assume (by replacing Ψ'' and Ψ by a suitable $\mathcal{O}_D^\times$-conjugate) that $T = L^\times$.

Since ξ is of depth zero, strictly less than r, Proposition 4.4 implies that the Weil representations corresponding to ϕ and $\xi\phi$ coincide; consequently $\widetilde{\rho}'' = \xi\widetilde{\rho}$ (where we have used that ξ is trivial on G_s to ensure this expression is independent of any choice of factorization of elements of TG_s), whence $\rho(\Psi'') = \xi\rho(\Psi)$.

Therefore by Mackey theory it suffices to show that for all $\gamma \in \mathcal{O}_D^\times$,

$$\mathrm{Hom}_{T_0G_s \cap {}^\gamma T_0G_s}(\xi\rho(\Psi), {}^\gamma\rho(\Psi)) = \{0\}. \tag{8.1}$$

This is clear for $\gamma \in T_0G_s$. Otherwise, we may WLOG scale γ by an element of $T_0 = \mathcal{O}_L^\times$ to assume $\gamma = 1 + vj$, with $v \in \mathcal{O}_L$, $\mathrm{val}_D(vj) < s$. Then one computes that an element $u \in \mathcal{O}_L^\times$ satisfies ${}^{\gamma^{-1}}u \in \mathcal{O}_L^\times G_s$ if and only if $\mathrm{val}_D(u-\overline{u}) \geq s - \mathrm{val}_D(vj) > 0$. It follows that

$$T_0G_s \cap {}^\gamma(T_0G_s) = \mathcal{O}_F^\times T_{s-\mathrm{val}_D(vj)}G_s \subseteq Z_0T_{0+}G_s.$$

Noting that $\rho(\Psi)$ and $\xi\rho(\Psi)$ agree on this subset, we have

$$\mathrm{Hom}_{T_0G_s \cap {}^\gamma T_0G_s}(\xi\rho(\Psi), {}^\gamma\rho(\Psi)) = \mathrm{Hom}_{T_0G_s \cap {}^\gamma T_0G_s}(\rho(\Psi), {}^\gamma\rho(\Psi)),$$

which must be zero, since the irreducibility of $\pi_{\mathcal{O}_D^\times}(\Psi)$ implies that only $\gamma \in T_0G_s$ can support an intertwining operator. Consequently (8.1) holds for all $\gamma \in \mathcal{O}_D^\times$, our contradiction. □

We deduce that the representations of degree greater than one are parametrized by $\mathcal{O}_D^\times$-conjugacy classes of $\mathcal{O}_D^\times$-data $\Psi_0 = (T_0, \phi_0, r, \chi_0)$ where these all represent the restriction of a $D^\times$-datum to $\mathcal{O}_D^\times$.

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