ON BRANCHING RULES OF DEPTH-ZERO REPRESENTATIONS

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Abstract. Using Bruhat-Tits theory, we analyse the restriction of depth-zero representations of a semisimple simply connected p-adic group G to a maximal compact subgroup K. We prove the coincidence of branching rules within classes of Deligne-Lusztig supercuspidal representations. Furthermore, we show that under mild compatibility conditions, the restriction to K of a Deligne-Lusztig supercuspidal representation of G intertwines with the restriction of a depth-zero principal series representation in infinitely many distinct components of arbitrarily large depth. Several qualitative results are also obtained, and their use and application is illustrated in an example.

1. Introduction

The branching rules considered here are those arising from the restriction of a complex admissible representation of a p-adic group G to a maximal compact open subgroup K. The ultimate goal of this analysis is to examine the interplay between the admissible duals of G and K, as well as to illuminate their respective structures. Aspects of this question for G include the theory of types and the study of newforms. On the other hand, the representation theory of K is still in its infancy, and branching rules provide a framework in which to search for results.

In this paper, we consider the restriction of certain depth-zero supercuspidal representations (those induced from inflations of Deligne-Lusztig cuspidal representations of associated finite groups of Lie type) to a hyperspecial maximal compact subgroup (denoted G_y) under the hypothesis that G is connected, simply connected, semisimple and split over a local non-archimedean ground field k of odd residual characteristic. Our particular focus is the set of "atypical" representations, that is, those which are not types, and are common to the branching rules of several representations. To this end we prove two main results.

The first concerns Deligne-Lusztig supercuspidal representations (recalled in Section 6). We parametrize their coarse decomposition into Mackey components by a set $X^+_{x,y}$ in Sections 4 and 5. In Theorem 6.4 we prove that whenever two Deligne-Lusztig supercuspidal representations arise from the same minisotropic torus and have the same central character, then a large portion of their branching rules are identical, namely, those parametrized by $int(X^+_{x,y})$. Moreover, in certain circumstances

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we prove that their complete restrictions to G_y coincide. This is Corollary 6.5; we illuminate its hypotheses with some examples.

The second main result, stated in Theorem 7.4, concerns the intertwining between restrictions of Deligne-Lusztig supercuspidal representations and principal series. We prove that the restriction of a Deligne-Lusztig supercuspidal representation π intertwines in infinitely many distinct components with any compatible depth-zero principal series representation $\mathrm{Ind}_B^G\chi$. The compatibility condition relates to the central character of the cuspidal representation inducing to π. Further refinements of this result, relating to the depths at which these intertwinings occur, are given as a sequence of corollaries in Section 7.

One of the main methods underlying the proofs of these results, beyond Mackey theory, is the analysis of subgroups of G which are stabilizers of subsets of an apartment $\mathcal{A}$. A result of independent interest is given in Proposition 3.3, where we relate certain stabilizer subgroups with Moy-Prasad filtration subgroups. We use this in Theorem 5.4 and Proposition 7.2 to glean information about the depths of the representations of G_y which arise.

Proving results on branching rules at this level of generality is a new and novel step, and anticipates the development of a general theory out of the case-by-case analysis achieved to date. In this sense the current work complements a series by the author on branching rules of $\mathrm{SL}(2,k)$ [11, 12, 13] and, with P. Campbell, $\mathrm{GL}(3,k)$ [2, 3]. Recently, U. Onn and P. Singla in [15] determined the complete decomposition into irreducible representations of the blocks of representations of [2]. We use their results in our example in Section 8 and anticipate that in fact the complete branching rules for $\mathrm{GL}(3,k)$ are attainable, using the above results and ideas inspired from the present paper.

The branching rules for $\mathrm{GL}(2,k)$ and $\mathrm{PGL}(2,k)$ were previously studied by W. Casselman, K. Hansen, A. Silberger and others. K. Maktouf and P. Torasso recently considered the branching rules of the Weil representation of a general symplectic group in [7], a particular case of which had been studied by D. Prasad in [16].

We assume that G is semisimple; this simplifies the exposition, particularly in Section 2. It is feasible and would be interesting to extend the results to G reductive, so that the Levi components of the proper parabolic subgroups are also in this class. This could allow an inductive analysis of branching rules including all parabolically induced representations.

Our proofs of the main theorems Theorem 6.4 and Theorem 7.4 rely on showing that certain double cosets support nonzero intertwining operators. These questions reduce to computations with Deligne-Lusztig characters. To determine which other double cosets also support intertwining operators would seem to require restricting representations to subgroups which are stabilizers of subsets of the Bruhat-Tits building not contained in any single apartment, and there is currently a dearth of literature on such subgroups. Moreover, the classification of the double coset spaces which arise is expected to be highly nontrivial: for $\mathrm{GL}(n,k)$, $n \geq 3$, it was shown by U. Onn, A. Prasad and L. Vaserstein in [14] to contain a wild classification problem in the limit.

The characters of Deligne-Lusztig cuspidal representations have a uniform description and are well-known; we make use of these in several computations. It would be useful to extend our results to other families of cuspidal representations: in fact for $\mathrm{SL}(2,k)$, the non-Deligne-Lusztig cuspidal representations give all atypical irreducible positive-depth components of all representations [13]. In general we expect they exhaust the atypical components of all supercuspidal representations.

An eventual goal is the complete decomposition of supercuspidal or principal series representations into irreducible G_y-representations. As we see in Section 5, this would imply describing the branching rules for the (simple) restriction of cuspidal representations to a parabolic subgroup. In Section 6 we relate this in the Deligne-Lusztig case to questions about the intersection of minisotropic tori with split Levi subgroups. These are interesting open problems in the representation theory of finite groups of Lie type which have been solved only in special cases using CHEVIE [5], for example.

Outline. In Section 2 we provide a survey of the background required, including several results from Bruhat-Tits theory. In Section 3 we present various properties of pointwise stabilizers of bounded subsets of an apartment, and prove that with few exceptions, the Moy-Prasad filtration subgroups are just stabilizer subgroups of certain convex subsets, up to a toral factor. Section 4 is devoted to determining a set of double coset representatives $X^+_{x,y}$ for the Mackey components of the supercuspidal representations of G for a special vertex y and any vertex x, and describing the structure of this set.

In Section 5 we prove general results about the restriction of any depth-zero supercuspidal representation of G to a (hyper)special maximal compact subgroup G_y. In Section 6 we specialize to the case of Deligne-Lusztig representations, proving the coincidence of their branching rules in many cases. We address principal series representations, proving their extensive intertwining over G_y with Deligne-Lusztig supercuspidal representations, in Section 7. We conclude in Section 8 with an example illustrating the use of the many related results in this paper for the group $G = \mathrm{SL}(3,k)$.

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2. Background: Summary

The main references for the background material in this section are [1, 20].

2.1. **Notation and conventions.** Let k be a local nonarchimedean field of residual characteristic $p \neq 2$. Its characteristic may be 0 or p. Its residue field κ is a finite field of order q. For the sake of brevity we will refer to our field as a p-adic field and our group as a p-adic group.

Let the integer ring of k be $\mathcal{R}$ and its maximal ideal $\mathcal{P}$. Let ϖ be a uniformizer, and normalize the valuation on k so that $\mathrm{val}(\varpi) = 1$. The units of $\mathcal{R}$ admit a filtration by subgroups U_n where $U_0 = \mathcal{R}^\times$ and $U_n = 1 + \mathcal{P}^n$ if $n > 0$.

Given a subgroup H of a group G we denote its center by $Z(H)$ and for any $g \in G$ write ${}^gH$ for the group gHg^{-1}. Whenever defined, a representation (σ, V) of H is smooth and V is a complex vector space. We write V^H for the fixed points of H on V. If $g \in G$ then we write ${}^g\sigma$ for the corresponding representation of ${}^gH$. Whenever defined, the group G acts on the normalized induced representation $\mathrm{Ind}_H^G\sigma$, or the compactly induced representation c-$\mathrm{Ind}_H^G\sigma$, by right translation.

Define $\widetilde{\mathbb{R}} = \mathbb{R} \cup (\mathbb{R}+) \cup \{\infty\}$ as in [1, 6.4.1]. For $r \in \mathbb{R}$ we denote by $\lceil r \rceil$ the least integer k satisfying $k \geq r$ and $\lceil r+ \rceil$ the least integer k with $k > r$. For $r \in \mathbb{R}$ we also set $\lfloor r \rfloor = -\lceil -r \rceil$.

2.2. **Structure theory.** Let $\mathbb{G}$ be a connected, simply connected, semisimple algebraic group which is defined and split over k. We write $G = \mathbb{G}(k)$. Let $\mathbb{S}$ be a maximal torus of $\mathbb{G}$, split over k, and denote the associated root system Φ. Choose positive roots $\Phi^+ \subset \Phi$ and simple roots $\Delta \subseteq \Phi^+$. Let $\mathbb{B}$ be the Borel subgroup of $\mathbb{G}$ defined by $(\mathbb{S}, \Phi^+)$ and $\mathbb{N}$ the normalizer of $\mathbb{S}$ in $\mathbb{G}$. We set $S = \mathbb{S}(k)$, $B = \mathbb{B}(k)$ and $N = \mathbb{N}(k)$. The corresponding finite Weyl group is $W_0 = N/S$.

Denote by $X_*(S) = \mathrm{Hom}_k(\mathbb{G}_m, \mathbb{S})$ the group of k-rational cocharacters of $\mathbb{S}$, and $X^*(S) = \mathrm{Hom}_k(\mathbb{S}, \mathbb{G}_m)$ the group of k-rational characters. Set $S_0 = \{t \in S \mid \forall \chi \in X^*(S), \mathrm{val}(\chi(t)) = 0\}$; this is the maximal compact subgroup of S.

For each $\alpha \in \Phi \subseteq X^*(S)$ we denote by $\alpha^\vee \in \Phi^\vee \subset X_*(S)$ the corresponding coroot. Since G is simply connected the lattice $X_*(S)$ is spanned by $\Phi^\vee$.

Denote by $\mathcal{A} = \mathcal{A}(\mathbb{G}, \mathbb{S}, k)$ the apartment corresponding to $(\mathbb{G}, \mathbb{S}, \Phi, k)$, which we think of as the affine space under $E = X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. The set of affine roots Φ_{af} is the set of affine functions $\{\alpha_k = \alpha + k \mid \alpha \in \Phi, k \in \mathbb{Z}\}$ on $\mathcal{A}$; α is the gradient of α_k. The set of hyperplanes $\{\beta = 0 \mid \beta \in \Phi_{af}\}$ define the walls of a polysimplicial complex structure on $\mathcal{A}$. Let D denote the *positive cone* $\{x \in \mathcal{A} \mid \forall \alpha \in \Phi^+, \alpha(x) > 0\}$ and let C, the *fundamental chamber*, be the unique chamber (also called alcove) in D containing $0 \in E$ in its closure.

The affine Weyl group W is generated by the affine reflections r_β for $\beta \in \Phi_{af}$, where r_β denotes the reflection in the hyperplane $\beta = 0$. Since $\mathbb{G}$ is simply connected, $W \cong X_*(S) \rtimes W_0$ and coincides with the extended affine Weyl group N/S_0. Here, W_0 acts as the stabilizer of $0 \in E$ and $X_*(S)$ acts by translations. For each $\ell \in X_*(S)$ let $t(\ell) \in W$ be its representative in W, which we identify with an element of $S \subset N$ when appropriate. For each $w \in W$ and $\ell \in X_*(S)$ we have $wt(\ell)w^{-1} = t(w\ell)$.

For any $x \in \mathcal{A}$, set $\Phi_x = \{\beta \in \Phi_{af} \mid \beta(x) = 0\}$ and $W_x = \langle r_\beta \in W \mid \beta \in \Phi_x \rangle$. Let Φ_x^{lin} be the set of gradients of elements of Φ_x; since G is split this is itself a root system. Choose a base Δ_x of Φ_x^{lin} so that the positive roots $\Phi_x^{lin,+}$ coincide with

$\Phi_x^{lin} \cap \Phi^+$. Let $W_x^{lin} \subset W_0$ be the subgroup generated by the linear reflections in elements of Φ_x^{lin}. Then the map of W_x^{lin} into W_x given by

$$w \mapsto t(x - wx)\, w = w\, t(w^{-1}x - x) \tag{2.1}$$

is a group isomorphism. If $W_x^{lin} = W_0$, then the point x is a vertex and is called a *special vertex* [20, 1.9]; not all vertices of $\mathcal{A}$ are special. Since $\mathbb{G}$ is split over k, x is special if and only if $\alpha(x) \in \mathbb{Z}$ for all $\alpha \in \Phi$.

2.3. **Filtrations and special subgroups.** Following [9], we associate to each $x \in \mathcal{A}, \alpha \in \Phi$ and $r \in \widetilde{\mathbb{R}}$ subgroups $\mathbb{G}_\alpha(k)_{x,r}$ of the corresponding root subgroup and, for $r \geq 0$, subgroups S_r of S. Then the *Moy-Prasad filtration group* for $r \in \widetilde{\mathbb{R}}_{\geq 0}$ is

$$G_{x,r} = \langle S_r, \mathbb{G}_\alpha(k)_{x,r} \mid \alpha \in \Phi \rangle.$$

By a choice of pinning we simply have $\mathbb{G}_\alpha(k)_{x,r} = \mathbb{G}_\alpha(\mathcal{P}^{\lceil r-\alpha(x)\rceil})$ and $S_r = \mathbb{S}(U_{\lceil r \rceil})$. Note that given $\ell \in X_*(S)$, we have

$${}^{t(\ell)}\mathbb{G}_\alpha(k)_{x,r} = \mathbb{G}_\alpha(k)_{x+\ell,r} = \mathbb{G}_\alpha(k)_{x,r-\alpha(\ell)}.$$

Let $\mathcal{B} = \mathcal{B}(\mathbb{G}, k)$ denote the (reduced) Bruhat-Tits building for $\mathbb{G}$ over k as in [1, 7.4.1]. Given any point $y \in \mathcal{B}$, there exist $g \in G$ and $x \in \mathcal{A}$ such that $y = g \cdot x$. For any $r \in \widetilde{\mathbb{R}}_{\geq 0}$, one defines $G_{y,r} := {}^gG_{x,r}$; this is independent of choices [9]. Since G is semisimple and simply connected, for any $x \in \mathcal{B}$, $G_{x,0}$ coincides with the stabilizer G_x of x in G [20, §3.1] and is the parahoric subgroup of G associated to x. If x is in an (open) alcove Γ then G_x is called an Iwahori subgroup.

In our setting, the maximal compact open subgroups of G are exactly the stabilizers of vertices of $\mathcal{B}$. If x is a special vertex, then G_x is a good maximal compact subgroup, in the sense that G admits decompositions $G = G_xSG_x$ (Cartan decomposition) and $G = G_xB$ (Iwasawa decomposition).

Given any $x \in \mathcal{B}$ the group $G_{x,+} := G_{x,0+}$ is the unipotent radical of the parahoric subgroup G_x. The quotient group $G_x/G_{x,+}$ is the group of κ-points of a connected reductive group $\mathbb{M}_x$ defined over κ (as in [9]). Set $\mathcal{S} := \mathbb{S}(\kappa) \subseteq \mathbb{M}_x(\kappa)$. If x is a hyperspecial vertex (as defined in [20, 1.10]) then $\mathbb{M}_x = \mathbb{G}$. Since in our setting G is split over k, hyperspecial vertices exist and coincide with the special vertices.

The maximal compact subgroups which are stabilizers of hyperspecial vertices are distinguished among all maximal compact subgroups in two ways. First, from their definition it follows that they are isomorphic to $\mathbb{G}(\mathcal{R})$. Secondly, they have maximal volume from among all maximal compact open subgroups [20, 3.8]. In this paper we choose to restrict to a maximal compact subgroup which is the stabilizer of a (hyper)special vertex, always denoted y.

To reduce notational burden, we write $\mathcal{G}_x = G_x/G_{x,+}$ for $\mathbb{M}_x(\kappa)$ and refer to parabolic subgroups ($\mathcal{P}$ and $\mathcal{B}$) and tori ($\mathcal{T}$) of $\mathcal{G}_x$ without reference to the algebraic group $\mathbb{M}_x$. This is unfortunate in one case arising in Section 6; let us define the needed terms here. Let $s \in \mathcal{G}_x$ be semisimple and let $\mathbb{C}_s$ denote its centralizer, which is a reductive subgroup of $\mathbb{M}_x$, and $\mathbb{C}_s^\circ$ its connected component subgroup. Then define $C^\circ_{\mathcal{G}_x}(s) = \mathbb{C}_s^\circ(\kappa)$. Note that if $s \in Z(\mathcal{G}_x)$ then $\mathbb{C}_s^\circ = \mathbb{M}_x$ and so $C^\circ_{\mathcal{G}_x}(s) = \mathcal{G}_x$.

2.4. **Representations of G.** Given an irreducible admissible representation π of G on a complex vector space V, the *depth* of π is a rational number defined as the least $r \in \mathbb{R}_{\geq 0}$ such that there exists $x \in \mathcal{B}(\mathbb{G}, k)$ for which V contains vectors invariant under $G_{x,r+}$ [9]. Where appropriate, we also refer to the depth of a representation of G_x, for fixed x. If x is a special vertex then the depth of any representation of G_x is a nonnegative integer.

By Jacquet's theorem, every irreducible admissible representation of G occurs as a subrepresentation of $\mathrm{Ind}_P^G\sigma$, for some parabolic subgroup P with Levi decomposition MN and supercuspidal representation σ of M (extended trivially across N). In case $P = B$, a Borel subgroup, the representation σ is simply a character χ of a split torus S and the representation $\mathrm{Ind}_B^G\chi$, which may fail to be irreducible, is called a principal series representation.

The classification of (irreducible) supercuspidal representations is not yet complete. It is a lasting conjecture, proven now in many cases, that all supercuspidal representations of depth r are compactly induced from a compact open subgroup. In case $r = 0$ this has been proven; more precisely L. Morris [8] and A. Moy and G. Prasad [10] proved that all depth-zero supercuspidal representations of G are given by

$$\pi = \text{c-}\mathrm{Ind}_{G_x}^G \tau \tag{2.2}$$

for some vertex $x \in \mathcal{B}$ and inflation τ of a cuspidal representation of $\mathcal{G}_x$. Among these cuspidal representations τ are the Deligne-Lusztig cuspidal representations, whose characters are well-known; see Section 6.

3. Stabilizers of subsets of $\mathcal{A}$

Let Ω be a bounded subset of $\mathcal{B}$. Its convex closure $\overline{\Omega}$ is the union of all the facets of $\mathcal{B}$ meeting Ω. The pointwise stabilizer of Ω is $G_\Omega = \cap_{x\in\Omega} G_x$ and it coincides with $G_{\overline{\Omega}}$ [1, Prop 2.4.13]. Given two points $x, y \in \mathcal{B}$, we have $G_x \cap G_y = G_{[x,y]}$, where $[x, y]$ is the unique geodesic joining x and y, which is a line in any apartment containing both points [1, Prop 2.5.4]. From these facts one concludes that if F is a facet such that $[x, y] \cap F \neq \emptyset$, then $G_{[x,y]} \subseteq G_F$.

F. Bruhat and J. Tits give the following description of G_Ω if $\Omega \subseteq \mathcal{A}$ [1, §6.4].

Proposition 3.1. *Suppose Ω is a bounded subset of $\mathcal{A}$. For each $\alpha \in \Phi$, define*

$$f_\Omega(\alpha) = \max\{\lceil -\alpha(x) \rceil \mid x \in \Omega\}.$$

Then $G_\Omega = S_0 U_\Omega$ where $U_\Omega = \langle \mathbb{G}_\alpha(\mathcal{P}^{f_\Omega(\alpha)}) \mid \alpha \in \Phi\rangle$. Furthermore, if Ω contains an open set of $\mathcal{A}$ then for any order on Φ the product map

$$S_0 \times \prod_{\alpha\in\Phi} \mathbb{G}_\alpha(\mathcal{P}^{f_\Omega(\alpha)}) \to G_\Omega \tag{3.1}$$

is a bijection.

As a particular consequence we note the following. Write $int(\Omega)$ for the interior of a set.

Corollary 3.2. *Let $\Omega \subset \mathcal{A}$ be a bounded set such that $x \in int(\Omega)$. Then in the factorization $G_\Omega = S_0 U_\Omega$ we have $U_\Omega \subseteq G_{x,+}$.*

Proof. Since $G_{x,+}$ is generated by S_1 and the groups $\mathbb{G}_\alpha(\mathcal{P}^{\lceil -\alpha(x)+\rceil})$, by Proposition 3.1 it suffices to show that for all $\alpha \in \Phi$, $f_\Omega(\alpha) > -\alpha(x)$. Since $x \in \Omega$, this is immediate if $\alpha(x) \notin \mathbb{Z}$. Otherwise, since $x \in int(\Omega)$ there exists some $z \in \Omega$ such that $\alpha(z) < \alpha(x)$, whence $f_\Omega(\alpha) \geq \lceil -\alpha(z) \rceil > -\alpha(x)$. □

We next wish to describe the relationship between subgroups G_Ω, with $\Omega \subseteq \mathcal{A}$, and Moy-Prasad filtration subgroups $G_{x,r}$. We begin by setting some notation.

Given an irreducible root system $\widetilde{\Phi}$ let $\widetilde{\Phi}^l$ denote the set of its long roots. If $\widetilde{\Phi}$ has two root lengths let $\widetilde{\Phi}^s = \widetilde{\Phi} \setminus \widetilde{\Phi}^l$ be its short roots; otherwise, set $\widetilde{\Phi}^s = \widetilde{\Phi}$. More generally, given a root system Φ with irreducible components $\widetilde{\Phi}_i$, for $1 \leq i \leq m$, define $\Phi^l = \cup_i \widetilde{\Phi}^l_i$ and $\Phi^s = \cup_i \widetilde{\Phi}^s_i$. Note that Φ, Φ^l and Φ^s all have the same rank.

Given $x \in \mathcal{A}$ and $r \in \mathbb{R}_{\geq 0}$, define

$$\Omega_x(\mathcal{A}, r) = \{z \in \mathcal{A} \mid \forall \alpha \in \Phi, |\alpha(x) - \alpha(z)| \leq r\}. \tag{3.2}$$

Define $\Omega^l_x(\mathcal{A}, r)$ and $\Omega^s_x(\mathcal{A}, r)$ by replacing Φ in (3.2) with Φ^l and Φ^s, respectively.

Proposition 3.3. *Let $x \in \mathcal{A}$ and $r \in \mathbb{R}_{\geq 0}$. Then*

$$G_{\Omega^s_x(\mathcal{A},r)} \subseteq S_0 G_{x,r} \subseteq G_{\Omega^l_x(\mathcal{A},r)} = G_{\Omega_x(\mathcal{A},r)}. \tag{3.3}$$

Moreover, whenever the root system Φ does not contain an irreducible component of type G_2 the second inclusion is an equality, that is, $S_0 G_{x,r} = G_{\Omega_x(\mathcal{A},r)}$.

Proof. First note that $\Omega^l_x(\mathcal{A}, r) = \Omega_x(\mathcal{A}, r)$. Namely, given $z \in \Omega^l_x(\mathcal{A}, r)$, choose a positive system $\Phi^{(+)}$ for which $z - x$ is in the closure of the positive cone and let $\theta^{(+)} \in \Phi^l$ be the corresponding highest (long) root. Then for each $\beta \in \Phi$, $|\beta(x - z)| \leq \theta^{(+)}(z - x) \leq r$, so $z \in \Omega_x(\mathcal{A}, r)$. Clearly also $\Omega^l_x(\mathcal{A}, r) \supseteq \Omega_x(\mathcal{A}, r)$. Hence $G_{\Omega^l_x(\mathcal{A},r)} = G_{\Omega_x(\mathcal{A},r)}$.

If $r = 0$ the groups appearing in (3.3) are all equal and there is nothing to show, so suppose $r > 0$. Each group is generated by S_0 and certain subgroups of the root groups; thus it suffices to show the inclusions on each root subgroup.

Let $z \in \Omega_x(\mathcal{A}, r)$. Then for each $\alpha \in \Phi$ we have $-\alpha(z) \leq r - \alpha(x)$, whence $\mathbb{G}_\alpha(\mathcal{P}^{\lceil r-\alpha(x)\rceil}) \subseteq \mathbb{G}_\alpha(\mathcal{P}^{\lceil -\alpha(z)\rceil})$. It follows that $G_{x,r} \subseteq \cap_{z \in \Omega_x(\mathcal{A},r)} G_z = G_{\Omega_x(\mathcal{A},r)}$, and the second inclusion holds.

Now consider the first inclusion. It suffices to show that for all $\alpha \in \Phi$ there exists $z_\alpha \in \Omega_x^s(\mathcal{A}, r)$ such that $-\alpha(z_\alpha) \geq r - \alpha(x)$.

First suppose $\alpha \in \Phi^s$. Then α lies in a unique irreducible component $\Phi^{s\prime}$ of Φ^s, corresponding to a subspace E' of $E = X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. Let Δ' be the base of $\Phi^{s\prime}$ with respect to which α is the highest root and let $H'_{\alpha,r}$ denote the (nonempty) intersection of the hyperplane $\alpha = r$ with the positive cone D' defined by Δ'. Choose $v \in H'_{\alpha,r} \subset E'$. Then for all $\beta \in \Phi^{s\prime}$ we have $|\beta(v)| \leq \alpha(v) = r$, and for all $\beta \in \Phi^s \setminus \Phi^{s\prime}$ we have $\beta(v) = 0 \leq r$. Therefore the element $z_\alpha = x - v$ satisfies our requirements.

Now let $\alpha \in \Phi \setminus \Phi^s$. Let Φ'' denote the irreducible component of Φ containing α, Δ'' the base with respect to which α is the highest root, and D'' the corresponding positive cone. Let $\alpha_0 \in \Phi^s \cap \Phi''$ be the corresponding highest short root and define $H'_{\alpha_0,r}$ as in the preceding paragraph. For any v in the nonempty intersection $H'_{\alpha_0,r} \cap D''$ we have $\alpha(v) \geq \alpha_0(v) = r$ and, as argued above, for all $\gamma \in \Phi^s$, $|\gamma(v)| \leq r$. Therefore $z_\alpha = x - v \in \Omega_x^s(\mathcal{A}, r)$ and satisfies $-\alpha(z_\alpha) \geq r - \alpha(x)$, as required.

Now consider the final assertion. If Φ is simply-laced then equality holds because $\Omega_x^s(\mathcal{A}, r) = \Omega_x(\mathcal{A}, r)$. Otherwise by the preceding arguments it suffices to show that in each non-simply-laced irreducible root system except G_2, there exists a short root α and a vector v such that $\alpha(v) = r$ and for all $\beta \neq \alpha$, $|\beta(v)| \leq r$. This is easily verified case-by-case. □

We remark that equality fails on the simple system of type G_2 because the boundary of $\Omega_x^s(\mathcal{A}, r)$ does not intersect the boundary of $\Omega_x(\mathcal{A}, r)$.

4. The double coset space $G_y \backslash G / G_x$

We begin by recalling a result about generalized BN-pairs [1, Proposition 7.4.15].

Proposition 4.1. *For $i = 1, 2$ let Ω_i denote a nonempty subset of $\mathcal{A}$, G_i its pointwise stabilizer in G, N_i the pointwise stabilizer of Ω_i in N, and $\widehat{W}_i$ its image in $W = N/S_0$. Then the natural map*

$$\widehat{W}_1 \backslash W / \widehat{W}_2 \to G_1 \backslash G / G_2$$

is bijective.

Corollary 4.2. *Let x, y be vertices of $\mathcal{A}$. Then*

$$G_y \backslash G / G_x \cong W_y \backslash W / W_x.$$

Proof. By Proposition 4.1, it suffices to note that for any vertex $z \in \mathcal{A}$, the group $\widehat{W}_z = (N \cap G_z)/S_0$ coincides with W_z, the group generated by the reflections in the affine hyperplanes through z. This follows in our case from [1, 7.1.3]. □

Let $D_x := \{z \in \mathcal{A} \mid \forall \alpha \in \Phi_x^{lin,+}, \alpha(z) > 0\}$ denote the positive cone for $\Phi_x^{lin,+}$.

Proposition 4.3. *Suppose y is special. A set of double coset representatives for $W_y\backslash W/W_x$ is given by*

$$\begin{aligned} X^+_{x,y} &= X_*(S) \cap (y - x + \overline{D_x}) \\ &= \{\ell \in X_*(S) \mid \forall \alpha \in \Phi_x^{lin,+}, \alpha(\ell) \geq \alpha(y-x)\}. \end{aligned}$$

Proof. Each $w \in W$ can be written uniquely as $w = w_0 t(v)$ for some $w_0 \in W_0$ and $v \in X_*(S)$. Since y is special, $w_0 \in W_y^{lin}$ and thus by (2.1) $w_y := w_0 t(w_0^{-1}y - y) \in W_y$. Therefore we can factor $w = w_y t(v_1)$ with $v_1 = v + y - w_0^{-1}y \in X_*(S)$ and it remains to show that there exists $\ell \in X^+_{x,y}$ such that $t(v_1) \in W_y t(\ell) W_x$.

Choose $w_1 \in W_x^{lin}$ such that $w_1(v_1 + x - y) \in \overline{D_x}$. Set $\ell := w_1(v_1 + x - y) + y - x = (y - w_1 y) + w_1 v_1 + (w_1 x - x)$. Since $w_1 \in W_x^{lin} \cap W_y^{lin}$, each summand lies in $X_*(S)$. Thus $\ell \in X^+_{x,y}$ and we have

$$\begin{aligned} t(\ell) &= t(y - w_1 y)\, t(w_1 v_1)\, t(w_1 x - x) \\ &= t(y - w_1 y)\, w_1 t(v_1) w_1^{-1}\, t(w_1 x - x) \qquad (4.1) \\ &= w'_y t(v_1) w_x \end{aligned}$$

where $w'_y := t(y - w_1 y) w_1 \in W_y$ and $w_x := w_1^{-1} t(w_1 x - x) \in W_x$. Hence $X^+_{x,y}$ exhausts $W_y\backslash W/W_x$.

Now suppose that $\ell, \ell' \in X^+_{x,y}$ are such that there exist $w_y \in W_y$ and $w_x \in W_x$ for which $t(\ell') = w_y t(\ell) w_x$. Since the composition is a translation, the linear parts of w_y and w_x are mutually inverse. Therefore there is some $w_0 \in W_x^{lin} \cap W_y^{lin} = W_x^{lin}$ such that $w_y = t(y - w_0 y) w_0$ and $w_x = w_0^{-1} t(w_0 x - x)$. Using (4.1) we conclude $\ell' = w_0(\ell + x - y) + (y - x)$. Since both $\ell + x - y$ and $\ell' + x - y = w_0(\ell + x - y)$ lie in $\overline{D_x}$ and are conjugate by W_x^{lin} they are equal. □

For example, if $y = x$ is special then $X^+_{y,y} = X_+$, the set of dominant cocharacters.

Remark 4.4. If $x \neq y$, then there is some $\alpha \in \Phi_x^{lin}$ for which $\alpha(x - y) \neq 0$, so that $X^+_{x,y} \neq X_+$. More generally $X^+_{x,y} = X_+ + (y - x)$ if and only if $x - y \in X_*(S)$, which will not arise if x, y are chosen in distinct orbits under G, for example.

Definition 4.5. Let $int(X^+_{x,y}) = X_*(S) \cap (y - x + D_x)$ and $\partial(X^+_{x,y}) = X^+_{x,y} \setminus int(X^+_{x,y})$, which we call the interior and the boundary of $X^+_{x,y}$, respectively.

We record some key properties of the interior of $X^+_{x,y}$ in two lemmas.

Lemma 4.6. *Let $\Upsilon_x = \{w \in W_0 \mid wD \subseteq D_x\}$. Then we have*

$$int(X^+_{x,y}) = \bigsqcup_{w \in \Upsilon_x} X^+_{x,y} \cap (y - x + wD).$$

Proof. Since $\Phi_x^{lin} \subseteq \Phi$, $\overline{D_x} = \cup_{w \in \Upsilon_x} w\overline{D}$ and thus $X^+_{x,y} \subset \cup_{w \in \Upsilon_x} \big(y - x + w\overline{D}\big)$. Fix $w \in \Upsilon_x$ and suppose $\ell \in X^+_{x,y} \cap (y - x + w(\overline{D} \setminus D))$. Then $x - y + \ell \in w(\overline{D} \setminus D)$ so there exists $\alpha \in \Phi$ such that $\alpha(x-y+\ell)=0$. But as y is special and $\ell \in X_*(S)$, this implies $\alpha(x) \in \mathbb{Z}$, whence $\alpha \in \Phi_x^{lin}$. Consequently, $x-y+\ell \in \partial(X_{x,y}^+)$. □

Lemma 4.7. *If $\ell \in int(X_{x,y}^+)$ then the convex closure of $[y, x+\ell]$ in $\mathcal{A}$ contains unique alcoves adjacent to each endpoint.*

Proof. For $z \in \{y, x+\ell\}$, let $\mathcal{F}_z$ be the set of facets of $\mathcal{A}$ containing z in their closure. A nontrivial line segment with an endpoint at z has nonzero intersection with a unique element F_z of $\mathcal{F}_z \setminus \{z\}$. We claim that in our case, F_z is an alcove. If not, then F_z, and consequently also $[y, x+\ell]$, is contained in the hyperplane $\alpha = k$ for some $\alpha \in \Phi$ and $k \in \mathbb{Z}$. In particular, we have $\alpha(x) = k - \alpha(\ell) \in \mathbb{Z}$ so $\alpha \in \Phi_x^{lin}$. But since $\alpha(y) = \alpha(x+\ell) = k$, we have $\alpha(\ell) = \alpha(y-x)$ whence $\ell \in \partial(X_{x,y}^+)$, a contradiction. □

5. Restrictions of Supercuspidal Representations to $\mathcal{G}_y$

For reference we cite a consequence of Mackey theory for compactly induced representations derived from [6].

Lemma 5.1. *Let G be the k-points of a linear algebraic group defined over k, with a compact open subgroup K and a compact-mod-center subgroup H. Let ρ be a smooth representation of H such that $\pi = c\text{-}\mathrm{Ind}_H^G \rho$ is admissible. For any $t \in K\backslash G/H$, the subspace of $c\text{-}\mathrm{Ind}_H^G\rho$ consisting of vectors supported on the double coset $Ht^{-1}K$ is K-invariant, and as a representation of K is isomorphic to $\mathrm{Ind}_{K\cap {}^tH}^K {}^t\sigma$. Thus we have*

$$\mathrm{Res}_K\, c\text{-}\mathrm{Ind}_H^G \sigma \cong \bigoplus_{t \in K\backslash G/H} \mathrm{Ind}_{K\cap {}^tH}^K {}^t\sigma. \tag{5.1}$$

In our case, let $H = G_x$ and $K = G_y$, for vertices $x, y \in \mathcal{A}$ with y special. Given an irreducible supercuspidal representation $\pi = \text{c-}\mathrm{Ind}_{G_x}^G \tau$ we therefore have

$$\begin{aligned}\mathrm{Res}_{G_y}\pi &= \mathrm{Res}_{G_y}\text{c-}\mathrm{Ind}_{G_x}^G \tau \\ &\cong \bigoplus_{t\in G_y\backslash G/G_x} \mathrm{Ind}_{G_y\cap {}^tG_x}^{G_y} {}^t\tau.\end{aligned}$$

By Proposition 4.3, we may choose the representatives of $G_y\backslash G/G_x$ to be $\{t(\ell) \mid \ell \in X_{x,y}^+\}$, whence $G_y \cap {}^{t(\ell)}G_x = G_y \cap G_{x+\ell} = G_{[y,x+\ell]}$. Thus we may rewrite the sum above as

$$\mathrm{Res}_{G_y}\pi \cong \bigoplus_{\ell \in X_{x,y}^+} \mathrm{Ind}_{G_{[y,x+\ell]}}^{G_y} {}^{t(\ell)}\tau. \tag{5.2}$$

We refer to the representation $\pi_\ell = \mathrm{Ind}_{G_{[y,x+\ell]}}^{G_y} {}^{t(\ell)}\tau$ as a *Mackey component* of $\mathrm{Res}_{G_y}\pi$. Note that this is not an irreducible representation in general.

Suppose from now on that τ has depth zero, and let us record some basic properties of the Mackey components π_ℓ.

Proposition 5.2. *Suppose* $\ell \in int(X^+_{x,y})$ *and set* $\pi_\ell = \mathrm{Ind}_{G_{[y,x+\ell]}}^{G_y} {}^{t(\ell)}\tau$. *Let* $\Phi^\dagger = \{\alpha \in \Phi \mid \alpha(\ell) > \alpha(y-x)\}$ *and set*

$$\eta(x-y+\ell) = \sum_{\alpha\in\Phi^\dagger} (\alpha(\ell) + \lceil \alpha(x-y)\rceil - 1).$$

Then

$$\deg(\pi_\ell) = \deg(\tau) q^{\eta(x-y+\ell)} \left|\mathcal{G}_y/\mathcal{B}\right|,$$

where $\mathcal{B}$ *is a Borel subgroup of* $\mathcal{G}_y$. *If* x *is also special then* $\eta(x-y+\ell) = 2\rho(x-y+\ell) - |\Phi^+|$, *where* $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$.

Proof. We suppose $\ell \in int(X^+_{x,y})$ and compute $[G_y \colon G_{[y,x+\ell]}]$.

By Lemma 4.7 the convex closure of $[y, x+\ell]$ contains an alcove Γ adjacent to y so $G_{[y,x+\ell]} \subseteq G_\Gamma \subseteq G_y$. Since G_Γ is an Iwahori subgroup of G contained in G_y, its image in $\mathcal{G}_y \cong \mathbb{G}(\kappa)$ is a Borel subgroup $\mathcal{B}$ of $\mathcal{G}_y$. This Borel subgroup is defined by a choice of positive system of Φ, namely the one consisting of the gradients of those affine roots in Φ_y which are positive on $x+\ell$. This is the set $\Phi^\dagger$.

We have $[G_y \colon G_\Gamma] = |\mathcal{G}_y/\mathcal{B}|$; note that this factor is independent of the choice of $\mathcal{B}$.

We use Proposition 3.1 to compute the remaining factor $[G_\Gamma \colon G_{[y,x+\ell]}]$. Set $\Omega = [y, x+\ell]$; then for each $\alpha \in \Phi$ we have $f_\Omega(\alpha) = \max\{-\alpha(y), \lceil -\alpha(x+\ell)\rceil\}$. If $\alpha \in \Phi^\dagger$ we have $f_\Omega(\alpha) = -\alpha(y) = f_\Gamma(\alpha)$ whereas $f_\Omega(-\alpha) = \lceil \alpha(x+\ell)\rceil$ and $f_\Gamma(-\alpha) = \alpha(y)+1$. It thus follows from (3.1) that

$$|G_\Gamma/G_{[y,x+\ell]}| = \prod_{\alpha\in\Phi^\dagger} q^{\lceil \alpha(x-y+\ell)-1\rceil} = q^{\eta(x-y+\ell)}. \tag{5.3}$$

If x is special, then $\alpha(x-y+\ell) \in \mathbb{Z}$ for all roots α and furthermore we deduce that $\Phi^\dagger = \Phi^+$. □

Remark 5.3. The computation of the degree of π_ℓ in the case that $\ell \in \partial(X^+_{x,y})$ is entirely analogous, using slightly more detailed results from [1, §6.4]. The factor $|\mathcal{G}_y/\mathcal{B}|$ is replaced by $|\mathcal{G}_y/\mathcal{P}|$ for a parabolic subgroup $\mathcal{P}$ of $\mathcal{G}_y$.

Theorem 5.4. *Let* $\ell \in X^+_{x,y}$. *Set*

$$r_0 = \max\{\beta(x-y+\ell) \mid \beta \in \Delta_x\}$$

and

$$s_0 = \max\{\lfloor \alpha(x-y+\ell)\rfloor \mid \alpha \in \Phi\}.$$

Then the depth d *of an irreducible subrepresentation of* $\mathrm{Ind}_{G_{[y,x+\ell]}}^{G_y} {}^{t(\ell)}\tau$ *satisfies* $r_0 \leq d \leq s_0$.

Proof. Let $\ell \in X^+_{x,y}$ and set $\pi_\ell = \mathrm{Ind}_{G_{[y,x+\ell]}}^{G_y} {}^{t(\ell)}\tau$. If the space of τ is denoted V_τ then the space of π_ℓ is $V_\ell = \{f \colon G_y \to V_\tau \mid \forall h \in G_{[y,x+\ell]}, \forall g \in G_y, f(hg) = {}^{t(\ell)}\tau(h)f(g)\}$. We prove that $V_\ell^{G_{y,r_0}} = \{0\}$ and $V_\ell^{G_{y,s_0+}} = V_\ell$, whence the result.

By construction, τ is trivial on $G_{x,+}$, and thus ${}^{t(\ell)}\tau$ is trivial on ${}^{t(\ell)}G_{x,+} = G_{x+\ell,+}$. Given a nonnegative integer s, the subgroup $G_{y,s+}$ is contained in $G_{x+\ell,+}$ if and only if for each $\alpha \in \Phi$, we have $\lceil (s - \alpha(y))+ \rceil \geq \lceil -\alpha(x+\ell)+ \rceil$. As $\alpha(y), \alpha(\ell) \in \mathbb{Z}$ this condition is equivalent to $s \geq \lfloor \alpha(y - x - \ell) \rfloor$. Set $s_0 = \max\{\lfloor \alpha(x - y + \ell) \rfloor \mid \alpha \in \Phi\}$; this is nonnegative since $\alpha(x - y + \ell) \geq 0$ for $\alpha \in \Phi_x^{lin,+}$. Thus G_{y,s_0+} is a normal subgroup of G_y contained in the kernel $G_{x+\ell,+}$ of ${}^{t(\ell)}\tau$, whence $V_\ell^{G_{y,s_0+}} = V_\ell$.

Now let $\mathcal{H}$ be the unipotent radical of a proper parabolic subgroup $\mathcal{P}$ of $\mathcal{G}_x$. Since τ is a cuspidal representation of the finite group $\mathcal{G}_x$, $V_\tau^{\mathcal{H}} = \{0\}$. Let $H \subseteq G_x$ be a subgroup satisfying $H/(H \cap G_{x,+}) = \mathcal{H}$. Using elementary arguments, and the normality of $G_{y,r}$ in G_y, one can show that if ${}^{t(\ell)}H \subseteq G_{y,r}$ then $V_\ell^{G_{y,r}} = \{0\}$.

Now each proper subset Δ' of Δ_x defines two proper parabolic subgroups of $\mathcal{G}_x$: the standard parabolic $\mathcal{P}_{\Delta'}$ and its opposite $\mathcal{P}_{\Delta'}^{op}$. Let $\mathcal{H}$ be the unipotent radical of $\mathcal{P}_{\Delta'}^{op}$. If Φ' is the subrootsystem of Φ_x^{lin} generated by Δ', then $\mathcal{H}$ is spanned by the root subgroups of $\mathcal{G}_x$ corresponding to $\{-\alpha \mid \alpha \in \widetilde{\Phi} = \Phi_x^{lin,+} \setminus \Phi'\}$. We may choose $H = \langle \mathbb{G}_{-\alpha}(k)_{x,0} \mid \alpha \in \widetilde{\Phi} \rangle \subseteq G_x$ as our lift of $\mathcal{H}$. Note that if x is not special then H is not necessarily contained in the unipotent radical of a parabolic subgroup of G.

We have ${}^{t(\ell)}H = \langle \mathbb{G}_{-\alpha}(k)_{x,\alpha(\ell)} \mid \alpha \in \widetilde{\Phi} \rangle$. Thus ${}^{t(\ell)}H \subseteq G_{y,r}$ if and only if for each $\alpha \in \widetilde{\Phi}$, $\lceil r + \alpha(y) \rceil \leq \lceil \alpha(x + \ell) \rceil$. Since α takes integral values on x, y and ℓ, this simplifies to $r \leq \alpha(x - y + \ell)$. Each simple root $\beta \in \Delta_x$ takes nonnegative values on $x - y + \ell$; by construction of $\widetilde{\Phi}$ we deduce that $\min\{\alpha(x - y + \ell) \mid \alpha \in \widetilde{\Phi}\}$ is attained on some simple root $\beta \in \Delta_x \setminus \Delta' \subseteq \widetilde{\Phi}$, whence we conclude $V_\ell^{G_{y,\beta(x-y+\ell)}} = \{0\}$.

Conversely, given $\beta \in \Delta_x$, choosing $\Delta' = \Delta_x \setminus \{\beta\}$ above ensures that $V_\ell^{G_{y,\beta(x-y+\ell)}} = \{0\}$. We conclude that $r_0 = \max\{\beta(x - y + \ell) \mid \beta \in \Delta_x\}$ has the property required. □

Example 1. For $G = \mathrm{SL}(2, k)$, with $y = 0$, one always has $r_0 = s_0$. Indeed, the depths of the irreducible components of π_ℓ were shown to be exactly $\delta(\ell) = \alpha(x - y + \ell) = x + \alpha(\ell)$ in [13, §5].

Example 2. For $G = \mathrm{Sp}(4, k)$, with $\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$, if x is the non-special vertex of C then $\Delta_x = \{\beta, 2\alpha + \beta\}$. Since the highest root of Φ is a simple root of Δ_x, the depth of each irreducible subrepresentation of π_ℓ is exactly $\max\{\beta(x - y + \ell), (2\alpha + \beta)(x - y + \ell)\} \in \mathbb{Z}$. If x is special, however, then the lower and upper bounds given in Theorem 5.4 cannot coincide.

More generally we note that if x is special, then $s_0 \leq h r_0$ where h is the height of the highest root of Φ.

Theorem 5.4 gives an immediate criterion for disjointness of representations of G_y occuring as factors of different Mackey components.

Corollary 5.5. *For $i = 1, 2$ let x_i be vertices of $\mathcal{A}$ and τ_i cuspidal representations of $\mathcal{G}_{x_i}$. Suppose $\ell_i \in X_{x_i,y}^+$ satisfy*

$$\max\{\alpha(x_1 - y + \ell_1) \mid \alpha \in \Phi\} < \max\{\alpha(x_2 - y + \ell_2) \mid \alpha \in \Delta_{x_2}\}.$$

Then the two Mackey components

$$\mathrm{Ind}_{G_{[y,x_1+\ell_1]}}^{G_y}{}^{t(\ell_1)}\tau_1 \quad \textit{and} \quad \mathrm{Ind}_{G_{[y,x_2+\ell_2]}}^{G_y}{}^{t(\ell_2)}\tau_2$$

are disjoint representations of G_y.

6. Case of Deligne-Lusztig cuspidal representations

Our main reference for this section is [4]. Recall that a minisotropic (maximal) torus $\mathcal{T}$ of $\mathcal{G}_x = G_x/G_{x,+}$ is one which is contained in no proper parabolic subgroup [18, II.1.11]. Writing $rk(\mathcal{H})$ for the k-rank of the group $\mathcal{H}$ we set $\varepsilon = (-1)^{rk(G)-rk(Z(\mathcal{G}_x))}$.

Let $\mathcal{T}$ be a minisotropic maximal torus of $\mathcal{G}_x$ and θ a character of $\mathcal{T}$. From this data P. Deligne and G. Lusztig constructed a virtual representation of $\mathcal{G}_x$ whose character we denote $R_{\mathcal{T}}^{\mathcal{G}_x}(\theta)$. If θ is in *general position* [4, §7.3], then $\varepsilon R_{\mathcal{T}}^{\mathcal{G}_x}(\theta)$ is irreducible and cuspidal, and the corresponding representation τ is called a Deligne-Lusztig cuspidal representation. This character is given on an element $h \in \mathcal{G}_x$ with Jordan decomposition $h = su$ by $\varepsilon R_{\mathcal{T}}^{\mathcal{G}_x}(\theta)(h) = 0$ if s is not conjugate of an element of $\mathcal{T}$, and otherwise, by

$$\varepsilon R_{\mathcal{T}}^{\mathcal{G}_x}(\theta)(h) = \frac{1}{|C^\circ_{\mathcal{G}_x}(s)|} \sum_{g\in\mathcal{G}_x, gsg^{-1}\in\mathcal{T}} \theta(gsg^{-1}) Q_{g^{-1}\mathcal{T}g}^{C^\circ_{\mathcal{G}_x}(s)}(u) \tag{6.1}$$

where $Q_{g^{-1}\mathcal{T}g}^{C^\circ_{\mathcal{G}_x}(s)}$ denotes the Green function, which takes values in $\mathbb{Z}$ [4, §7.6]. It is known that

$$\deg(\varepsilon R_{\mathcal{T}}^{\mathcal{G}_x}(\theta)) = Q_{\mathcal{T}}^{\mathcal{G}_x}(1) = \frac{|\mathcal{G}_x|}{|\mathcal{U}_x||\mathcal{T}|} \tag{6.2}$$

where $\mathcal{U}_x$ denotes the unipotent radical of a Borel subgroup $\mathcal{B}_x$ of $\mathcal{G}_x$.

Let us now work towards understanding the Mackey components of the corresponding supercuspidal representation $\pi = \text{c-Ind}_{G_x}^G \tau$. We begin with a general lemma.

Lemma 6.1. *Let τ be a depth-zero representation of G_x and $\ell \in X^+_{x,y}$. Let $F \neq \{x\}$ be the facet of $\mathcal{A}$ which contains x in its closure and meets $[y-\ell, x]$. Let $\mathcal{P} = G_F/G_{x,+}$ be the parabolic subgroup of $\mathcal{G}_x$ whose inflation to G_x is G_F. Then the irreducible components of $\mathrm{Res}_{G_{[y-\ell,x]}}\tau$ coincide with those of $\mathrm{Res}_{\mathcal{P}}\tau$.*

Proof. The existence of F follows as in the proof of Lemma 4.7. Since $x \in \overline{F}$ and F is a facet we have $G_{x,+} \subseteq G_F$ and $G_F/G_{x,+}$ is indeed a parabolic subgroup of $G_x/G_{x,+}$.

Let $\Omega \in \{F, [y-\ell, x]\}$. Since τ is the inflation of a representation, say for the moment $\overline{\tau}$, which is trivial on $G_{x,+}$, $\mathrm{Res}_{G_\Omega/(G_\Omega\cap G_{x,+})}\overline{\tau}$ and $\mathrm{Res}_{G_\Omega}\tau$ have the same irreducible components.

Now if β is an affine root such that $\beta(t) \geq 0$ for all $t \in \Omega$ and $\beta(x) > 0$ then necessarily $\beta(t) > 0$. It follows that the quotient $G_\Omega/(G_\Omega \cap G_{x,+})$ is uniquely determined by the set of affine roots vanishing on Ω. As these coincide for $\Omega = F$ and $\Omega = [y-\ell, x]$, the lemma follows. □

Thus to determine the decomposition into irreducible subrepresentations of each Mackey component, one should first determine the restriction of a cuspidal representation of $\mathcal{G}_x$ to a parabolic subgroup — a highly nontrivial open problem in general. Nevertheless, one can deduce some results in an important special case.

Let $\mathcal{B}_x = \mathcal{S}\mathcal{U}_x$ be a standard Borel subgroup of $\mathcal{G}_x$. Then the Jordan decomposition of any $h \in \mathcal{B}_x$ is $h = su$ with $s \in \mathcal{S}$ and $u \in \mathcal{U}_x$. But such an s is conjugate to an element of the minisotropic torus $\mathcal{T}$ if and only if $s \in Z(\mathcal{G}_x)$, since $\mathcal{T}$ cannot contain a split subtorus outside of the center. Consequently $C^\circ_{\mathcal{G}_x}(s) = \mathcal{G}_x$. The Green function depends only on the conjugacy class of $\mathcal{T}$ within $C^\circ_{\mathcal{G}_x}(s)$ and so in this case, is simply $Q^{\mathcal{G}_x}_{\mathcal{T}}$. Since s is central, $gsg^{-1} = s$, and the character formula from (6.1) simplifies to

$$\mathrm{Res}_{\mathcal{B}_x}\varepsilon R^{\mathcal{G}_x}_{\mathcal{T}}(\theta)(su) = \begin{cases} 0 & \text{if } s \notin Z(\mathcal{G}_x), \\ \theta(s)Q^{\mathcal{G}_x}_{\mathcal{T}}(u) & \text{otherwise.} \end{cases} \tag{6.3}$$

An immediate consequence of this calculation is the following proposition.

Lemma 6.2. *The restriction of a Deligne-Lusztig cuspidal representation $\varepsilon R^{\mathcal{G}_x}_{\mathcal{T}}(\theta)$ to a Borel subgroup of $\mathcal{G}_x$ depends only on the choice of minisotropic torus $\mathcal{T}$ and the restriction of θ to the center $Z(\mathcal{G}_x)$.*

Remark 6.3. In general we do not expect $\mathrm{Res}_{\mathcal{B}_x}\varepsilon R^{\mathcal{G}_x}_{\mathcal{T}}(\theta)$ to be irreducible; in fact we compute its self-intertwining number to be

$$\begin{aligned} \langle \varepsilon R^{\mathcal{G}_x}_{\mathcal{T}}(\theta), \varepsilon R^{\mathcal{G}_x}_{\mathcal{T}}(\theta)\rangle_{\mathcal{B}_x} &= \frac{1}{|\mathcal{B}_x|}\sum_{s\in Z(\mathcal{G}_x), u\in\mathcal{U}_x} |\theta(s)|^2 |Q^{\mathcal{G}_x}_{\mathcal{T}}(u)|^2 \\ &= \frac{|Z(\mathcal{G}_x)|}{|\mathcal{B}_x|}\sum_{u\in\mathcal{U}_x} Q^{\mathcal{G}_x}_{\mathcal{T}}(u)^2. \end{aligned}$$

Example 3. If $\mathcal{G}_x = \mathrm{SL}(2,\kappa)$ then we determined in [13] that this intertwining number is $2 = |Z(\mathcal{G}_x)|$.

Example 4. If $\mathcal{G}_x = \mathrm{SL}(3,\kappa)$ then we compute directly that $\sum_{u\in\mathcal{U}_x} Q^{\mathcal{G}_x}_{\mathcal{T}}(u)^2 = q^4(q-1)^2$ and hence that the intertwining number of $\mathrm{Res}_{\mathcal{B}_x} R^{\mathcal{G}_x}_{\mathcal{T}}(\theta)$ with itself is $|Z(\mathcal{G}_x)|q$, where $|Z(\mathcal{G}_x)| = 3$ if 3 divides $q-1$.

Theorem 6.4. *Let x, y be vertices of $\mathcal{A}$ with y special. Let τ_1 and τ_2 be two Deligne-Lusztig cuspidal representations of $\mathcal{G}_x$, induced from the same minisotropic torus and with the same central character. Let $\pi_i = c\text{-}\mathrm{Ind}^G_{G_x}\tau_i$ be the corresponding depth-zero supercuspidal representations of G, for $i = 1, 2$. Then for each $\ell \in int(X^+_{x,y})$, the Mackey components of $\mathrm{Res}_{G_y}\pi_i$ corresponding to ℓ coincide for $i = 1, 2$. That is, we have*

$$\mathrm{Ind}^{G_y}_{G_{[y,x+\ell]}}{}^{t(\ell)}\tau_1 \cong \mathrm{Ind}^{G_y}_{G_{[y,x+\ell]}}{}^{t(\ell)}\tau_2. \tag{6.4}$$

Proof. The induced representation $\mathrm{Ind}_{G_{[y,x+\ell]}}^{G_y} {}^{t(\ell)}\tau_i$ is determined by $\mathrm{Res}_{G_{[y,x+\ell]}} {}^{t(\ell)}\tau_i$. Conjugating by $t(\ell)^{-1}$ we deduce that (6.4) would follow from

$$\mathrm{Res}_{G_{[y-\ell,x]}}\tau_1 \cong \mathrm{Res}_{G_{[y-\ell,x]}}\tau_2. \tag{6.5}$$

By Lemma 4.7, the geodesic $[y, x+\ell]$ meets a unique alcove Γ' adjacent to $x+\ell$; thus $\Gamma = \Gamma' - \ell$ is an alcove adjacent to x meeting $[y-\ell, x]$. It follows that the group $G_\Gamma/G_{x,+}$ is a Borel subgroup $\mathcal{B}_x$ of $\mathcal{G}_x$ and thus by Lemma 6.1, (6.5) is equivalent to the condition that $\mathrm{Res}_{\mathcal{B}_x}\tau_1 \cong \mathrm{Res}_{\mathcal{B}_x}\tau_2$; this follows from Lemma 6.2 by our hypotheses on τ_1 and τ_2. □

We can say much more, under certain circumstances.

Corollary 6.5. *Suppose we are in the setting of Theorem 6.4 and suppose additionally that $\mathcal{T}$ has the property that $\mathcal{T} \cap \mathcal{P} = Z(\mathcal{G}_x)$ for all proper parabolic subgroups $\mathcal{P}$ of $\mathcal{G}_x$. Then if y and x are not conjugate under G we have*

$$\mathrm{Res}_{G_y}\pi_1 \cong \mathrm{Res}_{G_y}\pi_2,$$

whereas if $y = x$ then there exists a representation W of G_y such that we can write

$$\mathrm{Res}_{G_y}\pi_i \cong \tau_i \oplus W$$

for $i = 1, 2$, with W common to both.

Proof. Under the given hypotheses, the method of the proof of Lemma 6.2 applies equally to the restriction of τ_i to any proper parabolic subgroup $\mathcal{P}$, and we conclude that $\mathrm{Res}_{\mathcal{P}}\tau_1 \cong \mathrm{Res}_{\mathcal{P}}\tau_2$. Therefore, for each $\ell \in X^+_{x,y}$ for which $y - \ell \neq x$, we may apply Lemma 6.1 to conclude that $\mathrm{Res}_{G_{[y-\ell,x]}}\tau_1 \cong \mathrm{Res}_{G_{[y-\ell,x]}}\tau_2$ and hence that the corresponding Mackey components coincide, that is, for all such ℓ, (6.4) holds. If x and y lie in distinct orbits under W, then $y - \ell \neq x$ holds for all $\ell \in X^+_{x,y}$. Otherwise, we may without loss of generality assume $y = x$, in which case the single non-shared Mackey component is simply

$$\mathrm{Ind}_{G_y}^{G_y}\tau_i = \tau_i.$$

The result follows. □

We conclude with some examples to explore the hypotheses of Corollary 6.5.

Example 5. The group $\mathcal{G} = \mathrm{SL}(3,\kappa)$ has a unique maximal anisotropic torus $\mathcal{T}$ of order q^2+q+1 [18, II.1.10]. Up to conjugacy there is only one proper parabolic subgroup which is not a Borel, and its Levi component L is isomorphic to $\mathrm{GL}(2,\kappa)$. Were $t \in \mathcal{T} \cap L$, it would be a semisimple element of L, hence lie in a torus of $\mathrm{GL}(2,\kappa)$. However these tori have order $(q-1)^2$ and q^2-1; in each case the gcd with q^2+q+1 is 3 and thus the order of t is either 1 or 3. Hence $t \in Z(\mathrm{SL}(3,\kappa))$ and the hypotheses of Corollary 6.5 hold.

Example 6. Let $\mathcal{G} = \mathrm{Sp}(4,\kappa)$. Then $\mathcal{G}$ has two maximal anisotropic tori up to conjugacy [17, II.4–8]. The Coxeter torus T_{w_0} has order q^2+1 and one concludes as in Example 5 that it cannot meet a proper parabolic subgroup except in the center of $\mathcal{G}$.

The other anisotropic torus T_{-1} corresponds to the element $w = -1$ in the Weyl group. In [19] this torus is the subgroup $H_4 = \langle a_4 \rangle \times \langle b_4 \rangle$, which is isomorphic to $N_1(\kappa')^2$ where $N_1(\kappa')$ is the group of norm-one elements of a quadratic extension κ' of κ. One can show that the generator a_4 lies in a parabolic subgroup with Levi component isomorphic to $\mathrm{SL}(2,\kappa) \times \mathrm{GL}(1,\kappa)$.

Therefore the hypotheses of Corollary 6.5 hold for $\mathcal{T} = T_{w_0}$ but not for $\mathcal{T} = T_{-1}$.

7. Intertwining with principal series

Let χ be a depth-zero character of S. Construct the parabolically induced representation $\mathrm{Ind}_B^G\chi$; this is a depth-zero (possibly reducible) principal series representation of G. We denote by V the space of $\mathrm{Ind}_B^G\chi$.

Let y be a special vertex. Then as $G = BG_y$ we have

$$\mathrm{Res}_{G_y}\mathrm{Ind}_B^G\chi \cong \mathrm{Ind}_{B\cap G_y}^{G_y}\chi_0 \tag{7.1}$$

where χ_0 denotes the restriction of χ to $B\cap G_y$. Note that twisting χ by any unramified character produces the same restriction to G_y; this holds in particular for the modular character which appears in our normalized induction $\mathrm{Ind}_B^G\chi$.

7.1. **Some subrepresentations.** The nature of parabolic induction is such that it is easier to construct a filtration of V by G_y-invariant subspaces than a direct sum decomposition.

Lemma 7.1. *Let Ω be a bounded convex closed subset of $\mathcal{A}$ satisfying $\overline{C} \subseteq \Omega \subseteq \overline{D}$. Then χ_0 extends trivially to a character of $G_{y+\Omega}$ and $\mathrm{Ind}_{G_{y+\Omega}}^{G_y}\chi_0$ is a subrepresentation of $\mathrm{Ind}_{B\cap G_y}^{G_y}\chi_0$. Let $V_{y+\Omega}$ denote the space of this representation; it is finite-dimensional. If $\Omega' \supseteq \Omega$ is another such set, then $V_{y+\Omega'} \supseteq V_{y+\Omega}$.*

Proof. By Proposition 3.1, we can write G_{y+C} as S_0U_{y+C} where S_0 normalizes U_{y+C}. Since χ_0 is trivial on $S_0\cap U_{y+C} = S_1$, it extends to a character of G_{y+C}, trivial on U_{y+C}, which coincides with χ_0 on $B\cap G_y$. Denote again by χ_0 the restriction of this character to any subgroup of G_{y+C}. Since $\overline{C} \subseteq \Omega \subset \overline{D}$, we have $B \cap G_y \subset G_{y+\Omega} \subseteq G_{y+C}$. The rest follows. □

We have the following estimates relating to the depth and degree of $V_{y+\Omega}$.

Proposition 7.2. *Suppose n is a positive integer. If $y + \Omega \subseteq \Omega_y(\mathcal{A}, n)$ then $V_{y+\Omega} \subseteq V^{G_{y,n}}$ and the depth of any irreducible subrepresentation of $V_{y+\Omega}$ is strictly less than n. Moreover, $\dim(V^{G_{y,n}}) = |\mathcal{G}_y/\mathcal{B}|\; q^{(n-1)|\Phi^+|}$ for any Borel subgroup $\mathcal{B}$ of $\mathcal{G}_y$.*

Proof. We may restrict to integral values since y is special. If $y + \Omega \subseteq \Omega_y(\mathcal{A}, n)$ then $G_{y,n} \subseteq G_{y+\Omega}$ by Proposition 3.3. Since $n > 0$ we further have $G_{y,n} \subseteq S_1U_{y+\Omega} = \ker(\chi_0)$, so it acts trivially on the induced representation, yielding $V_{y+\Omega}^{G_{y,n}} = V_{y+\Omega}$.

In fact, this defines an isomorphism $V^{G_{y,n}} \cong \mathrm{Ind}_{(B\cap G_y)G_{y,n}}^{G_y}\chi_0$, whence the dimension formula. □

Remark 7.3. Let $r \in \mathbb{R}_{>0}$. If Φ does not contain an irreducible component of type G_2 then from Proposition 3.3 we may deduce that $(B \cap G_y)G_{y,r} = G_{\Omega_r}$ where $\Omega_r = \overline{D} \cap \Omega_y(\mathcal{A}, r)$. In general, however, the partially ordered filtration of subrepresentations $V_{y+\Omega}$ does not necessarily include the subrepresentations $V^{G_{y,r}}$ of $G_{y,r}$-fixed vectors. Although not needed here, note that one can obtain a much finer filtration (which in particular includes the $V^{G_{y,r}}$) by replacing the subgroups $G_{y+\Omega}$ with groups G_f where f is a concave function [1, §6.4] satisfying $f(\alpha) = -\alpha(y)$ and $f(-\alpha) > \alpha(y)$ for all $\alpha \in \Phi^+$, as in [2].

7.2. **Calculations on intertwining.** Now let $\pi = \text{c-Ind}_{G_x}^G\tau$ be a depth-zero supercuspidal representation of G. Let $\ell \in X_{x,y}^+$ and denote the corresponding Mackey component by $\pi_\ell = \mathrm{Ind}_{G_{[y,x+\ell]}}^{G_y}{}^{t(\ell)}\tau$.

Then for each set Ω as in Lemma 7.1, we have

$$\begin{aligned} \mathrm{Hom}_{G_y}(\pi_\ell, \mathrm{Ind}_{G_{y+\Omega}}^{G_y}\chi_0) &\cong \mathrm{Hom}_{G_{[y,x+\ell]}}({}^{t(\ell)}\tau, \mathrm{Res}_{G_{[y,x+\ell]}}\mathrm{Ind}_{G_{y+\Omega}}^{G_y}\chi_0) \\ &\cong \mathrm{Hom}_{G_{[y,x+\ell]}}({}^{t(\ell)}\tau, \oplus_{c\in\Psi_{x,y,\Omega}}\mathrm{Ind}_{G_{[y,x+\ell]}\cap {}^cG_{y+\Omega}}^{G_{[y,x+\ell]}}{}^c\chi_0) \\ &\cong \oplus_{c\in\Psi_{x,y,\Omega}}\mathrm{Hom}_{G_{[y,x+\ell]}\cap {}^cG_{y+\Omega}}({}^{t(\ell)}\tau, {}^c\chi_0) \\ &\cong \oplus_{c\in\Psi_{x,y,\Omega}}\mathrm{Hom}_{G_{[x,y-\ell]}\cap G_{t(-\ell)c\cdot(y+\Omega)}}(\tau, {}^{t(-\ell)c}\chi_0) \end{aligned} \tag{7.2}$$

where $\Psi_{x,y,\Omega} = G_{[y,x+\ell]}\backslash G_y/G_{y+\Omega}$.

Determining a set of representatives for $\Psi_{x,y,\Omega}$ is a large subset of the problem of classifying $B \cap G_y$ double cosets in G_y, which for some groups is known to contain the matrix pair problem, that is, be wild [14]. Furthermore, while $t(-\ell)c\cdot(y+\Omega)$ will be a convex closed subset of an apartment $\mathcal{A}'$, meeting $\mathcal{A}$ in at least the point $y - \ell$, it is not to be expected that there exists a choice of such $\mathcal{A}'$ which also contains x. Thus in general the convex closure of $[x, y - \ell] \cup t(-\ell)c \cdot (y + \Omega)$ is not contained in any apartment of $\mathcal{B}$, and therefore its stabilizer is much more difficult to describe.

Nevertheless, there remain some tractable cases to consider, which suffice for proving the following theorem. Let $Z_x \subseteq G_x$ denote the full preimage of $Z(\mathcal{G}_x) \subseteq \mathcal{G}_x$.

Theorem 7.4. *Let τ be a Deligne-Lusztig cuspidal representation of $\mathcal{G}_x$ with central character θ. Let $\widehat{\theta}$ denote the inflation of θ to Z_x. Let χ be a character of S such that for some $w \in W_0$, $\mathrm{Res}_{Z_x}{}^w\chi = \widehat{\theta}$. Then the restrictions to G_y of*

$$\pi^s = c\text{-}\mathrm{Ind}_{G_x}^G\tau \quad \textit{and} \quad \pi^p = \mathrm{Ind}_B^G\chi$$

have infinitely many distinct irreducible representations in common, of arbitrarily large depth.

7.3. **Proof of Theorem 7.4.** We begin by proving that each Mackey component of a Deligne-Lusztig supercuspidal representation corresponding to an element of $int(X_{x,y}^+)$ intertwines with any compatible principal series representation.

Proposition 7.5. *Let τ, θ and $\widehat{\theta}$ be as above. Let $\ell \in int(X^+_{x,y})$ and define $w \in \Upsilon_x$ by $\ell + x - y \in wD$. Let χ be a character of S such that $\mathrm{Res}_{Z_x}{}^w\chi = \widehat{\theta}$ and denote by χ_0 the trivial extension of χ to any subgroup of G_{y+C}. Then the representations*

$$\mathrm{Ind}_{G_{[y,x+\ell]}}^{G_y} {}^{t_\ell}\tau \quad and \quad \mathrm{Ind}_{G_{y+\Omega}}^{G_y} \chi_0$$

intertwine, for all bounded convex closed subsets Ω with $\overline{C} \subseteq \Omega \subset \overline{D}$ for which $x - y + \ell \in w\Omega$.

Proof. Note that as $S_1 = {}^{w^{-1}}S_1 \subseteq Z_x$, the hypotheses imply that χ has depth zero. It therefore suffices to show that there exists a nonzero summand in (7.2).

The existence and uniqueness of $w \in \Upsilon_x \subseteq W_0$ follows from Lemma 4.6. By (2.1), $w_y := t(y - wy)w \in W_y$, which we lift to an element of G_y. Set $\Omega' = t(-\ell)w_y \cdot (y + \Omega) = y - \ell + w\Omega$. When $x - y + \ell \in w\Omega$ both x and $y - \ell$ lie in Ω', so $G_{[x,y-\ell]} \cap G_{\Omega'} = G_{\Omega'}$. Defining $U_{\Omega'}$ as in Proposition 3.1, we deduce that the summand for $c = w_y$ in (7.2) is

$$\mathrm{Hom}_{S_0 U_{\Omega'}}(\tau, {}^{t(-\ell)w_y}\chi_0). \tag{7.3}$$

By hypothesis we have $x \in int(\Omega')$ so by Corollary 3.2, $U_{\Omega'} \subseteq G_{x,+} \subseteq \ker(\tau)$. On the other hand, note that

$${}^{w_y^{-1}t(\ell)}(S_0 U_{\Omega'}) = S_0 U_{w_y^{-1}t(\ell)\cdot\Omega'} = S_0 U_{y+\Omega}$$

and that χ_0 was defined to be trivial on $U_{y+\Omega}$. Therefore ${}^{t(-\ell)w_y}\chi_0$ is trivial on $U_{\Omega'}$. Moreover, on S_0 the character ${}^{t(-\ell)w_y}\chi_0$ coincides with ${}^w\chi$. Thus (7.3) is isomorphic to

$$\mathrm{Hom}_{S_0}(\tau, {}^w\chi).$$

Using the character formula from (6.3), the intertwining of the character $\varepsilon R_{\mathcal{T}}^{\mathcal{G}_x}(\theta)$ of τ with ${}^w\chi$ is given on S_0 by

$$\begin{aligned}
\langle \varepsilon R_{\mathcal{T}}^{\mathcal{G}_x}(\theta), {}^w\chi\rangle_{S_0} &= \frac{1}{|S_0|}\int_{S_0} \varepsilon R_{\mathcal{T}}^{\mathcal{G}_x}(\theta)(s)\overline{{}^w\chi(s)}\, ds \\
&= \frac{1}{|S_0|}\int_{Z(\mathcal{G}_x)}\int_{S_1} \deg(\tau)\theta(z)\overline{{}^w\chi(zs_1)}\, dz ds_1 \\
&= \begin{cases} 0 & \text{if } \mathrm{Res}_{Z_x}{}^w\chi \neq \widehat{\theta} \\ \deg(\tau)\frac{|Z(\mathcal{G}_x)|}{|S|} & \text{otherwise.}\end{cases}
\end{aligned}$$

Consequently $\mathrm{Hom}_{S_0}(\tau, {}^w\chi) \neq \{0\}$ exactly when the restriction of ${}^w\chi$ to Z_x coincides with $\widehat{\theta}$. The proposition follows. □

We now do away with the apparent dependence on w in Proposition 7.5.

Corollary 7.6. *Let $\mathrm{Ind}_B^G\chi$ be a depth-zero principal series representation. Suppose τ is a Deligne-Lusztig cuspidal representation of $\mathcal{G}_x$ with central character θ with inflation $\widehat{\theta}$ to Z_x. Let $w \in W_0$ and suppose $\mathrm{Res}_{Z_x}{}^w\chi = \widehat{\theta}$. Then for every $\ell \in int(X^+_{x,y})$,* *there exists a subrepresentation of the Mackey component* π_ℓ *of* $\mathrm{Res}_{G_y} c\text{-}\mathrm{Ind}_{G_x}^G \tau$ *which is isomorphic to a subrepresentation of* $\mathrm{Res}_{G_y}\mathrm{Ind}_B^G \chi$.

Proof. For any $\ell \in int(X^+_{x,y})$, we define $w_0 \in \Upsilon_x$ as in Proposition 7.5. Thus $x - y + \ell \in w_0 D$. Choose a bounded closed convex set Ω satisfying $C \cup \{w_0^{-1}(x-y+\ell)\} \subset \Omega \subset \overline{D}$. Since $\mathrm{Res}_{Z_x}({}^{w_0}({}^{w_0^{-1}w}\chi)) = \mathrm{Res}_{Z_x}{}^{w}\chi = \widehat{\theta}$, Proposition 7.5 implies that π_ℓ intertwines with the subrepresentation of $\mathrm{Res}_{G_y}\mathrm{Ind}_B^G({}^{w_0^{-1}w}\chi)$ induced from $G_{y+\Omega}$. Consequently $\mathrm{Res}_{G_y}\mathrm{Ind}_B^G({}^{w_0^{-1}w}\chi)$ contains a subrepresentation of G_y which is isomorphic to a subrepresentation of π_ℓ. Finally, since $w_0^{-1}w \in W_0$, $\mathrm{Ind}_B^G({}^{w_0^{-1}w}\chi) \cong \mathrm{Ind}_B^G\chi$ as representations of G and therefore their restrictions to G_y must also be isomorphic. □

Although the subrepresentations arising in Corollary 7.6 are not necessarily distinct, we have the following result.

Corollary 7.7. *Let* π^s *be a Deligne-Lusztig supercuspidal representation and* π^p *a depth-zero principal series representation, which are compatible in the sense of Corollary 7.6. Then* $\mathrm{Res}_{G_y}\pi^s$ *and* $\mathrm{Res}_{G_y}\pi^p$ *have infinitely many distinct components in common, and the set of depths of these components is unbounded.*

Proof. The first part follows from Corollary 7.6 by the Pigeonhole Principle since there are infinitely many $\ell \in int(X^+_{x,y})$ and the admissibility of each supercuspidal representation implies each G_y-subrepresentation occurs with finite multiplicity.

More explicitly, we may restrict ℓ to an infinite subset of $X^+_{x,y} \cap (y - x + D)$ in which every pair of elements satisfy the conditions of Corollary 5.5 thereby ensuring that their components are distinct. By Theorem 5.4, the set of depths of these representations is unbounded above. □

Remark 7.8. Given a depth-zero principal series representation, one may ask if for each vertex x and minisotropic maximal torus $\mathcal{T} \subseteq \mathcal{G}_x$ there exists a Deligne-Lusztig cuspidal character $R_{\mathcal{T}}^{\mathcal{G}_x}(\theta)$ such that the corresponding supercuspidal representation is compatible with χ. This is equivalent to the question of the existence of a character θ of $\mathcal{T}$, coinciding with χ on $Z(\mathcal{G}_x)$, and which is in general position, that is, not fixed by any nontrivial element of W_x. For q sufficiently large, this follows from the arguments in [4, Lemma 8.4.2] with minor modification.

8. An example

We now illustrate the use of the results of Sections 5 to 7 with an example.

Let $G = \mathrm{SL}(3,k)$. Suppose that $p \neq 3$ and $3 \nmid (q-1)$, whence we have simply $\mathrm{GL}(3,\mathcal{R}) = Z(\mathrm{GL}(3,\mathcal{R}))\mathrm{SL}(3,\mathcal{R})$ and the irreducible representations of $\mathrm{GL}(3,\mathcal{R})$ and $\mathrm{SL}(3,\mathcal{R})$ coincide. Since all vertices of $\mathcal{B}$ are special and are conjugate by $\mathrm{GL}(3,\mathcal{R})$, we may without loss of generality set $x = y = 0$.

For ease of notation, let G_{abc} denote the subgroup which is the intersection with G of the set of matrices of the form $\left[\begin{smallmatrix} \mathcal{R} & \mathcal{R} & \mathcal{R} \\ \mathcal{P}^a & \mathcal{R} & \mathcal{R} \\ \mathcal{P}^c & \mathcal{P}^b & \mathcal{R} \end{smallmatrix}\right]$.

Since $Z(\mathrm{SL}(3,\kappa)) = \{1\}$ and there is a unique anisotropic torus in $\mathrm{SL}(3,\kappa)$, the compatibility condition in Theorem 6.4 trivially holds for any two Deligne-Lusztig cuspidal representations of $\mathrm{SL}(3,\kappa)$. Furthermore, as noted in Example 5, the hypotheses of Corollary 6.5 hold, implying that all the components of positive depth in the restriction to $\mathrm{SL}(3,\mathcal{R})$ of any two such supercuspidal representations coincide. So let us fix one choice of Deligne-Lusztig cuspidal representation τ and set $\pi = \text{c-Ind}_{G_y}^G \tau$.

We next fix one Mackey component π_ℓ and determine its decomposition into irreducible representations of G_0.

Let $\Delta = \{\alpha, \beta\}$; then $X_{x,y}^+ = X_+ = \{\ell \in X_*(S) \mid \alpha(\ell) \geq 0 \text{ and } \beta(\ell) \geq 0\}$. The smallest nonzero element in X_+ is $\ell = (\alpha+\beta)^\vee$; it lies in $int(X_+)$. We have $G_{[0,\ell]} = G_{112}$. By Theorem 5.4, the depths of the irreducible subrepresentations of π_ℓ are either 1 or 2. By Corollary 5.5, π_ℓ is disjoint from every other $\pi_{\ell'}$ except possibly $\pi_{2\ell}$: if $\ell' \in X_+ \setminus \{0, \ell, 2\ell\}$ then its irreducible subrepresentations have depth at least 3. The degree of τ is $(q-1)(q^2-1)$ from (6.2), so by Proposition 5.2 the degree of π_ℓ is $q(q+1)(q^2-1)(q^3-1)$.

By Remark 6.3 and as computed in Example 4, the intertwining number of $\mathrm{Res}_{\mathcal{B}^{op}}\tau$ with itself is q. To decompose it into irreducible subrepresentations, we begin by restricting τ to the unipotent radical $\mathcal{U}^{op}$ of $\mathcal{B}^{op}$, which is simply a Heisenberg group over κ with center $\mathbb{G}_{-\alpha-\beta}(\kappa)$. Using character computations, one determines that the restriction of τ to $\mathcal{U}^{op}$ consists of $(q-1)$ copies of each of the $(q-1)$ distinct Stone-Von Neumann representations H_ψ (corresponding to a nontrivial central character ψ) together with the $(q-1)^2$ characters of $\mathcal{U}^{op}$ arising from the characters $\psi_{-\alpha} \otimes \psi_{-\beta} \otimes 1$ of $\mathbb{G}_{-\alpha}(\kappa) \times \mathbb{G}_{-\beta}(\kappa) \times \mathbb{G}_{-\alpha-\beta}(\kappa)$ where neither $\psi_{-\alpha}$ nor $\psi_{-\beta}$ is trivial. It is then straightforward to determine that $\mathrm{Res}_{\mathcal{B}^{op}}\tau$ decomposes as q distinct irreducible representations: the $q-1$ components of $\rho = \mathrm{Ind}_{\mathcal{U}^{op}}^{\mathcal{B}^{op}} H_\psi$, each of dimension $q(q-1)$, and the representation $\phi = \mathrm{Ind}_{\mathcal{U}^{op}}^{\mathcal{B}^{op}} \psi_{-\alpha} \otimes \psi_{-\beta} \otimes 1$ (for any choice of nontrivial $\psi_{-\alpha}$ and $\psi_{-\beta}$), of dimension $(q-1)^2$.

We obtain a corresponding decomposition $\pi_\ell = \rho' \oplus \phi'$, where $\rho' := \mathrm{Ind}_{G_{[0,\ell]}}^{G_0}{}^{t(\ell)}\rho$ and $\phi' = \mathrm{Ind}_{G_{[0,\ell]}}^{G_0}{}^{t(\ell)}\phi$. One can show this induction is irreducible (that is, ϕ' is irreducible and ρ' has exactly $q-1$ irreducible components) by computing directly that of the seven double cosets of $G_{[0,\ell]}$ in G_0, only the trivial one supports intertwining operators.

Now let us consider the intertwining of π_ℓ with a principal series representation.

Since $Z(\mathcal{G}_0) = \{1\}$, the compatibility condition of Theorem 7.4 holds for any depth zero character of S_0, so without loss of generality let $\chi = 1$ and consider $\mathrm{Ind}_B^G 1$. By Proposition 7.2, all intertwining of π_ℓ with $V = \mathrm{Ind}_{B\cap G_0}^{G_0} 1$ must already occur with the $q^6(q^2+1+1)(q+1)$-dimensional subrepresentation $V^{G_{0,3}} = V_{\Omega_3} = \mathrm{Ind}_{G_{\Omega_3}}^{G_0} 1$ where $\Omega_n = \overline{D} \cap \Omega_0(\mathcal{A}, n)$.

By the proof of Proposition 7.5, the intertwining number of π_ℓ with V_{Ω_3} is at least $\deg(\tau)/|\mathcal{S}| = q+1$, which suggests the possibility that π_ℓ can be embedded into V as a subrepresentation. This is in fact the case, as follows.

Using the same arguments as in the proof of Proposition 7.5, one can show directly that the irreducible representation ϕ' intertwines already with V_{Ω_2}; in particular this implies that ϕ' has depth 1, which follows readily from its construction. Furthermore, we can compute that this intertwining occurs for no larger subgroup than G_{Ω_2}, whence it lives on the highest-dimensional quotient of V_{Ω_2}. By [2], this component is irreducible of dimension $q(q^2-1)(q^3-1)$, whence another proof of the irreducibility of ϕ'.

On the other hand, ρ' has no intertwining on the identity double coset with V_{Ω_2}, but the remaining q intertwining operators map to $V_{223} = \mathrm{Ind}_{G_{223}}^{G_0} 1$. In [2] it is shown that V_{223} contains two isomorphic irreducible representations $W_{123} \cong W_{213}$ of dimension $q^2(q+1)(q^3-1)$ and in [15] it is shown that the quotient W_{223} of V_{223} by the sum of all its proper subrepresentations V_{ijk} decomposes into a direct sum of $q-2$ distinct irreducible representations of this same degree, each distinct from W_{123}. Moreover, these exhaust all irreducible subrepresentations of V_{223} of this degree. We deduce that ρ' coincides with $W_{123} \oplus W_{223} \cong W_{213} \oplus W_{223} \subseteq V$.

Consequently (*cf.* Corollary 7.6) π_ℓ embeds in V (nonuniquely!). Furthermore, in this case all the irreducible components of π_ℓ of positive depth are atypical, in the sense that they occur also in principal series representations. However, there exist several irreducible components of depth 1 in $\mathrm{Ind}_{B\cap G_0}^{G_0} 1$ [2] which do not occur in π_ℓ, and therefore by our analysis we may conclude that they are not atypical with any Deligne-Lusztig supercuspidal representation.

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