Optimal synthesis for a minimum time problem in the plane with a triangle control set

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Abstract

This work is devoted to the study of a minimum time control problem where the state is governed by a two-dimensional affine system with two inputs taking values within a triangle, and describing a series of two interconnected chemostats. We show the existence of a subset of the invariant domain $\mathcal{D}$ associated to the system such that if the target is in this set, then it can be reached by any initial condition in $\mathcal{D}$. For every target point in this subset, we provide an optimal synthesis of the problem by decomposing $\mathcal{D}$ into two subsets. In the first one, we give an explicit expression of the value function, and we show that there exist infinitely many optimal solutions whereas in the second one, we show that the optimal strategy is of singular type.

Keywords. optimal control, minimal time problem, Pontryagin maximum principle, chemostat

MSC. 49J15, 49K15, 49N25.

1 Introduction

Cascade of chemostats or bioreactors is quite popular in microbiology (called "gradostats" [17, 32, 8, 30]) or in bioprocesses (called "serial tanks" [5]), because it is a way to create a gradient of resources (see also [7, 9]). Such gradients are expected to be more realistic to mimic real environment for studying micro-organisms growth [14, 6]. In the biotechnological industry, series of bioreactors are also known to be more efficient for the resource conversion than single tanks [15, 5, 10, 22].

In order to better understand the alcoholic fermentation process, several models have been proposed, which typically rely on the interconnection of chemostats (see e.g. [4]). The choice of the input flow rate in order to drive the system to a desired target value (typically an equilibrium of the system) in a minimal amount of time plays a key role in these studies, and finding an adequate feeding strategy can significally reduce the cost of such an operation (see e.g. [1, 12, 24] for the design of optimal feedback controls laws in the setting of fed-batch bioreactors).

The model that we consider in this work relies on the following system describing a series of two chemostats which means that the input substrate concentration in the second reactor is exactly the substrate concentration in the first one:

$$\begin{cases} \dot{x}_1 = [\mu(s_1) - u_1]x_1, \\ \dot{s}_1 = -\mu(s_1)x_1 + u_1(s_{in} - s_1), \\ \dot{x}_2 = [\mu(s_2) - u_2]x_2, \\ \dot{s}_2 = -\mu(s_2)x_2 + u_2(s_1 - s_2). \end{cases} \tag{1.1}$$

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Here, x_1 (resp. x_2) is the concentration of biomass in the first (resp. second) reactor, s_1 (resp. s_2) is the substrate concentration in the first (resp. second) reactor, s_{in} is the input substrate concentration in the first reactor, u_1 and u_2 are the dilution rates in the two reactors, and μ is the growth function describing how the species grows on the substrate (typically of Monod type, see [23, 31]). As both chemostats are in series, the dilution rate of the second reactor is less than the first one, which means that both controls u_1 and u_2 take values within the set:

$$U := \{v = (v_1, v_2) \in \mathbb{R}^2 \mid 0 \leq v_2 \leq v_1 \leq u_{max}\}, \tag{1.2}$$

where u_{max} is the maximal dilution rate. It is well known that the set $\mathcal{V} := \{(x, s) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \mid x + s = s_{in}\}$ is invariant and attractive for the sub-system (x_1, s_1) of (1.1), see [31]. Also, for practical operations, it is interesting to drive (1.1) from an equilibrium point (defined by the choice of a constant control) to another one in a minimal amount of time. So, in this work we assume that initial conditions for the substrate s_1 and s_2 are in $\mathcal{V}$, which implies $x_1 + s_1 = x_2 + s_2 = s_{in}$. It follows that (1.1) can be put into a two-dimensional affine system with two inputs:

$$\begin{cases} \dot{s}_1 = -\mu(s_1)(s_{in} - s_1) + u_1(s_{in} - s_1), \\ \dot{s}_2 = -\mu(s_2)(s_{in} - s_2) + u_2(s_1 - s_2). \end{cases} \tag{1.3}$$

Initial conditions for (1.3) are taken in the set $\mathcal{D}$ defined by:

$$\mathcal{D} := \{(s_1, s_2) \in \mathbb{R}_+^2 \mid 0 \leq s_1 \leq s_2 \leq s_{in}\}, \tag{1.4}$$

and from (1.2), we define the admissible control set $\mathcal{U}$ by:

$$\mathcal{U} := \{u = (u_1, u_2) : [0, \infty) \to U \mid \text{meas.}\}. \tag{1.5}$$

The optimal control problem that we consider reads as follows. Our aim is to find an optimal feedback control law $u \in \mathcal{U}$ steering (1.3) from a given initial condition $s_0 = (s_1^0, s_2^0) \in \mathcal{D}$ to a fixed target point $\bar{s} = (\bar{s}_1, \bar{s}_2) \in \mathcal{D}$ in a minimal amount of time:

$$\inf_{u \in \mathcal{U}} t_f(u) \text{ s.t. } s(t_f(u)) = \bar{s}, \tag{1.6}$$

where $s(\cdot)$ is the unique solution of (1.3) with control $u \in \mathcal{U}$, and $t_f(u) \in [0, +\infty)$ is the time to steer the solution of $s(\cdot)$ to the target $\bar{s}$.

From the constraint (1.2), we understand the difficulty of the controllability of (1.3). A consequence of (1.2) is that the switching functions associated to u_1 and u_2 provided by Pontryagin maximum principle are non independent. In particular, we can show that singular arcs occur only if the switching function associated to u_2 is vanishing implying $u_1 = u_2$. Also, one can see that (1.2) implies that there are some points $s_0 \in \mathcal{D}$ which are non-locally controllable for any time $t > 0$ in the sense given by [29].

Before investigating the minimum time problem, we prove the existence of a subset $\Delta_0^- \subset \mathcal{D}$ such that if the target is in Δ_0^-, then it can be reached by any initial condition in $\mathcal{D}$. For the minimum time problem where the state is governed by a two-dimensional system with a single input u:

$$\dot{s} = f(s) + ug(s), \; |u| \leq 1, \tag{1.7}$$

with $f, g : \mathbb{R}^2 \to \mathbb{R}^2$, the following assumptions are often required in order to ensure the local controllability of the target point (see e.g. [3, 20]) :

- (i) the target point satisfies $f(\bar{s}) = 0$, and the matrix $[g(\bar{s}), Df(\bar{s})g(\bar{s})]$ is of full rank.
- (ii) the target point satisfies $f(\bar{s}) \neq 0$ and there exist u such that $f(\bar{s}) + ug(\bar{s}) = 0$ and $|u| \leq 1$.

In our setting, we define exactly Δ_0^- as the set of points satisfying $\det(f(\bar{s}), g(\bar{s})) < 0$ (whenever (1.3) is put into (1.7) with $u_1 = u_2$), and we prove that this condition implies an analogous condition as condition (ii) mentioned above (see section 2).

Now, given a target point in Δ_0^-, we first solve (1.6) for initial conditions which are above the two trajectories satisfying (1.3) backward in time with $u = (0, 0)$ and $u = (1, 1)$, and starting from $\bar{s}$. We provide in this case an explicit expression of the value function of the optimal control problem. In this case, there exist infinitely many optimal trajectories connecting a point to the target. For initial conditions below the two curves mentioned above, we first show that an optimal control necessarily satisfies $u_1 = u_2$. However finding an issue to the minimal time problem in this case is more delicate. Indeed, we can show the existence of attractive *singular steady-state points* (see [3]) which are exactly the points at the intersection of a singular arc and the set of points where f and g are collinear. In the case of an attractive singular steady-state point, a singular arc cannot reach this point in finite time.

We provide in this setting a complete description of optimal trajectories under the additional assumption that this point is in the domain above the two curves mentioned above. We then prove that a singular arc strategy (which roughly speaking consists in the most rapid approach to the singular arc) is optimal (see e.g. [24, 12]).

The paper is organized as follows. In section 2, we review controllability properties of (1.3) that will be useful in order to provide an optimal synthesis of the problem, and we define the set Δ_0^-. In section 3, we prove an optimality result in a sub-domain of $\mathcal{D}$, and in section 4 we provide an optimal feedback control on the complementary in $\mathcal{D}$ of this set.

2 Controllability properties

In this section, we introduce a set $\mathcal{C}(\overline{s}) \subset \mathcal{D}$, depending on the target point $\overline{s} \in \text{int } \mathcal{D}$, such that $\overline{s}$ is reachable from any initial condition in $\mathcal{C}(\overline{s})$. Moreover, we show the existence of a subset of $\mathcal{D}$, Δ_0^-, such that if the target is in this set, then it is reachable form any initial condition in $\mathcal{D}$.

2.1 Invariance of $\mathcal{D}$

Without any loss of generality, we may assume that $u_{max} = 1$. We will also suppose that the growth function $\mu : [0, +\infty[\to \mathbb{R}$ satisfies the following hypotheses:

(H1) $\mu(\cdot)$ is of class C^2 on $[0, +\infty[$,

(H2) $\mu(s) < 1$, for all $s \in [0, s_{in}[$ and $\mu(0) = 0$,

(H3) $\mu(\cdot)$ is increasing on $[0, s_{\text{in}}]$.

These assumptions are satisfied by the Monod growth function (see [23]).

Remark 2.1. *The second assumption means that the maximum dilution rate is large enough in order to compete the growth of the species on the substrate. It is essential for studying the controllability of* (1.3).

By (H1), one has that (1.3) satisfies the standard hypotheses of the Cauchy-Lipschitz Theorem for the existence and uniqueness of solutions of (1.3). Given an initial condition $s_0 = (s_1^0, s_2^0) \in \mathcal{D}$ and a control $u \in \mathcal{U}$, we define $s(\cdot, s_0, u)$ as the unique solution of (1.3) defined on a maximal time interval $[0, t_f)$, $t_f > 0$, associated to the control u and with initial condition s_0 at time 0. For $t \geq 0$, we also denote by $s(-t, s_0, u)$ (with slight abuse of notation in view of the definition of $t \longmapsto s(t, s_0, u)$ for $t \geq 0$) the unique solution of the system:

$$\begin{cases} \dot{s}_1 = \mu(s_1)(s_{in} - s_1) - u_1(s_{in} - s_1), \\ \dot{s}_2 = \mu(s_2)(s_{in} - s_2) - u_2(s_1 - s_2), \end{cases} \tag{2.1}$$

defined on a maximal time interval $[0, t_f)$, $t_f > 0$ associated to the control u and with initial condition s_0 at time 0. When there is no ambiguity on the control, we also write $t \longmapsto \tilde{s}(t)$ in place of $s(-t, s_0, u)$. Equivalently, $t \longmapsto \tilde{s}(t)$ is the solution of (1.3) *backward in time* with control u and starting at time 0 with initial condition s_0. The next lemma is a simple consequence of Gronwall's Lemma.

Lemma 2.1. *The set $\mathcal{D}$ is invariant by system* (1.3).

Proof. Let $s_0 = (s_1^0, s_2^0) \in \mathcal{D}$, $u \in \mathcal{U}$ and $(s_1(\cdot), s_2(\cdot)) := s(\cdot, s_0, u)$. Let $t_f > 0$ the maximal time where s is defined. First notice that the point $(s_{\text{in}}, s_{\text{in}})$ is an equilibrium of (1.3). Therefore by uniqueness of the solution and by the inequality $s_1^0 \leq s_{\text{in}}$, one has that $s_1(t) \leq s_{\text{in}}$, for all $t \in [0, t_f)$. Let us show that $s_2(t) \leq s_1(t)$ for all $t \in [0, t_f)$. Set $z := s_1 - s_2$. Whenever $z = 0$, we have:

$$\dot{z} = u_1(s_{\text{in}} - s_1) \geq 0, \tag{2.2}$$

Thus, as $s_2(0) \leq s_1(0)$, one obtains that $s_2(t) \leq s_1(t)$ for all $t \in [0, t_f)$. Also, when $s_1 = 0$ one has:

$$\dot{s_1} = u_1 s_{\mathrm{in}} \geq 0, \tag{2.3}$$

so, as $s_1(0) \geq 0$, one has $s_1(t) \geq 0$ for all t. Finally, we have $s_2(t) \geq 0$ for all $t \in [0, t_f)$. Indeed, when $s_2 = 0$, we get:

$$\dot{s_2} = u_2 s_1 \geq 0, \tag{2.4}$$

as $s_1 \geq 0$. Therefore, as $s_2(0) \geq 0$, one has $s_2(t) \geq 0$ for all $t \in [0, t_f)$. This shows that solutions of (1.3) starting at time 0 in $\mathcal{D}$ are defined on $[0, +\infty)$, and that $s(t, s_0, u) \in \mathcal{D}$ for all $t \geq 0$. This concludes the proof. □

Remark 2.2. *Given $s_0 \in \mathcal{D}$ such that $\overline{s}$ is reachable from s_0, we can prove the existence of an optimal control for* (1.6) *as follows. In view of the regularity of the dynamics (see (H2)) and using that* (1.3) *is affine with respect to u and that U is compact, we are in position to apply Fillipov's Theorem (see e.g. [20]).*

2.2 Definition of $\mathcal{C}(\overline{s})$

Given a parametrized continuous curve $p = (p_1, p_2) : [0, t_f) \to \mathcal{D}$ defined on some time interval $[0, t_f)$, we denote by $p_{\geq}$ its epigraph:

$$p_{\geq} := \{(s_1, s_2) \in \mathcal{D} \mid \exists\, t \in [0, t_f) \text{ s.t. } s_1 = p_1(t) \text{ and } s_2 \geq p_2(t)\}.$$

In what follows, we set up a parametrized curve with increasing first component. First, we study the solution of (1.3) backward in time with $u_1 = u_2 = 1$ (see Fig. 1).

Lemma 2.2. *Let $\overline{s} = (\overline{s_1}, \overline{s_2}) \in int\ \mathcal{D}$, fix the control $u := (1, 1)$, and let $t_f > 0$ the maximal time interval where $\tilde{s}$ (with $u_1 = u_2 = 1$) is defined. Then, the set*

$$\{t \in (0, t_f) \mid (\tilde{s}_1(t), \tilde{s}_2(t)) \notin \mathcal{D}\}, \tag{2.5}$$

is non-empty and if $\tilde{t}$ is the exit time of $\mathcal{D}$ defined by

$$\tilde{t} := \inf\{t \in (0, t_f) \mid (\tilde{s}_1(t), \tilde{s}_2(t)) \notin \mathcal{D}\}, \tag{2.6}$$

then, we have either $\tilde{s}_2(\tilde{t}) = 0$ or $\tilde{s}_1(\tilde{t}) = \tilde{s}_2(\tilde{t})$. Moreover, $\tilde{s}_1(\cdot)$ is decreasing on $[0, \tilde{t}]$.

Proof. We first show that $\tilde{s}_1(\cdot)$ is decreasing on $[0, t_f[$. As s_{in} is an equilibrium of $\dot{\tilde{s}}_1 = (\mu(\tilde{s}_1) - 1)(s_{\mathrm{in}} - \tilde{s}_1)$, together with $\overline{s_1} < s_{\mathrm{in}}$, one obtains that $\tilde{s}_1(t) < s_{\mathrm{in}}$ for all $t \in [0, t_f[$. Therefore by Hypothesis (H2), one has $\dot{\tilde{s}}_1(t) < 0$ for almost every $t \in [0, t_f[$ implying that $\tilde{s}_1(\cdot)$ is decreasing on $[0, t_f[$. If there exists t such that $\tilde{s}_2(t) = \tilde{s}_1(t)$, the result is proved. So, we assume that $\tilde{s}_2(t) < \tilde{s}_1(t)$ for all $t \in [0, t_f[$. Suppose that $\tilde{s}_2(t) \geq 0$ for all $t \in [0, t_f)$. It follows that $\tilde{s}_1$ is lower bounder and decreasing, so it converges to some $\tilde{s}_1^\infty$ when t tends to t_f. We necessarily have $\tilde{s}_1^\infty = s_{\mathrm{in}}$ (the only equilibrium with $u = (1, 1)$) which is a contradiction. Thus, if $\tilde{t}$ is defined by (2.6), we obtain by continuity of $(\tilde{s}_1(\cdot), \tilde{s}_2(\cdot))$ that $(\tilde{s}_1(\tilde{t}), \tilde{s}_2(\tilde{t})) \in \partial\mathcal{D}$ which implies the result as $\tilde{s}_1(\tilde{t}) = s_{\mathrm{in}}$ is impossible by monotonicity of $\tilde{s}_1(\cdot)$. □

We now study the solution of (1.3) backward in time with $u_1 = u_2 = 0$ (see Fig. 1).

Proposition 2.1. *Let $\overline{s} = (\overline{s_1}, \overline{s_2}) \in int\ \mathcal{D}$ and fix the control $u = (0, 0)$. Then, $\tilde{s}$ (with $u_1 = u_2 = 0$) satisfies:*
(i) The function $\tilde{s}$ is defined over $[0, +\infty)$, moreover $\tilde{s}_i(\cdot)$ is increasing on $[0, +\infty[$, $i \in \{1, 2\}$ and:

$$\lim_{t \to +\infty} \tilde{s}(t) = (s_{in}, s_{in}). \tag{2.7}$$

(ii) When t goes to infinity, the ratio $\frac{s_{in} - \tilde{s}_2}{s_{in} - \tilde{s}_1}$ converges to a finite value.
(iii) If $\gamma : s \longmapsto \mu(s)(s_{in} - s)$ is concave over $[0, s_{in}]$, then there exists a convex function $\sigma_1 \longmapsto \sigma_2 = h(\sigma_1)$ defined over $[0, s_{in})$, of class C^2, and such that $\sigma_2 = h(\sigma_1)$ if and only if there exists $t \geq 0$ such that $(\sigma_1, \sigma_2) = (\tilde{s}_1(t), \tilde{s}_2(t))$.

Proof. We remark that $\tilde{s}_1(\cdot)$ and $\tilde{s}_2(\cdot)$ are both solutions of:

$$\dot{\sigma} = \mu(\sigma)(s_{\rm in} - \sigma). \tag{2.8}$$

Moreover, as 0 and $s_{\rm in}$ are the only equilibria of (2.8), and as $(\overline{s}_1, \overline{s}_2) \in \operatorname{int} \mathcal{D}$ implies the inequality $0 < \overline{s}_2 < \overline{s}_1 < s_{\rm in}$, we obtain that for each $i \in \{1, 2\}$, $\tilde{s}_i(\cdot)$ is defined over $[0, +\infty[$ and that

$$0 < \tilde{s}_2(t) < \tilde{s}_1(t) < s_{\rm in}, \quad \forall t \in [0, +\infty[. \tag{2.9}$$

On the other hand, (2.9) together with (2.8) implies that $\tilde{s}_i(\cdot)$ is increasing on $[0, +\infty[$ and upper-bounded. Therefore $\tilde{s}_i(\cdot)$ converges to some $\tilde{s}_i^\infty$ when t tends to $+\infty$, and we have $\tilde{s}_i^\infty = s_{\rm in}$ by monotonicity of $\tilde{s}_i(\cdot)$.

To prove (ii), notice from (2.8) that $\tilde{s}_1$ and $\tilde{s}_2$ satisfy the equality:

$$\int_{\tilde{s}_2}^{\tilde{s}_1} \frac{d\sigma}{\mu(\sigma)(s_{in} - \sigma)} = \int_{\overline{s}_2}^{\overline{s}_1} \frac{d\sigma}{\mu(\sigma)(s_{in} - \sigma)}.$$

Let $\overline{C} > 0$ denotes the right member of the previous display. Now, we can write:

$$\begin{aligned} \overline{C} = \int_{\tilde{s}_2}^{\tilde{s}_1} \frac{d\sigma}{\mu(\sigma)(s_{in} - \sigma)} &= \int_{s_{in}-\tilde{s}_1}^{s_{in}-\tilde{s}_2} \frac{d\sigma}{\mu(s_{in} - \sigma)\sigma} \\ &= \int_{s_{in}-\tilde{s}_1}^{s_{in}-\tilde{s}_2} \frac{d\sigma}{\sigma\mu(s_{in})} + \int_{s_{in}-\tilde{s}_1}^{s_{in}-\tilde{s}_2} \left[\frac{1}{\mu(s_{in} - \sigma)\sigma} - \frac{1}{\sigma\mu(s_{in})}\right] d\sigma \\ &= \frac{1}{\mu(s_{in})} \ln\left(\frac{s_{in} - \tilde{s}_2}{s_{in} - \tilde{s}_1}\right) + I(\tilde{s}_1, \tilde{s}_2), \end{aligned} \tag{2.10}$$

where $I(\tilde{s}_1, \tilde{s}_2)$ is the second integral of (2.10). Using that $s_{in} - \tilde{s}_1$ and $s_{in} - \tilde{s}_2$ are bounded, we deduce from Taylor inequality that $I(\tilde{s}_1, \tilde{s}_2)$ is bounded on the time interval $[0, +\infty)$. It follows that the ratio $z := \frac{s_{in}-\tilde{s}_2}{s_{in}-\tilde{s}_1}$ is bounded on this interval. Finally, notice that

$$\dot{z} = z[\mu(\tilde{s}_1) - \mu(\tilde{s}_2)],$$

so that z is bounded and strictly increasing. Hence, it converges to a finite value as was to be proved.

To prove (iii), notice from (i) and hypothesis (H1) that $t \longmapsto \tilde{s}_1(t)$ defines a diffeomorphism of class C^2 from $[0, +\infty)$ into $[\overline{s}_1, s_{in})$. By composition, it follows that the mapping $\sigma_1 \in [\overline{s}_1, s_{in}) \longmapsto \sigma_2$ where $\sigma_2 = \tilde{s}_2(t)$ is the graph of a function h of class C^2. Moreover, we have:

$$h'(\sigma_1) = \frac{d\sigma_2}{d\sigma_1} = \frac{\gamma(\tilde{s}_2)}{\gamma(\tilde{s}_1)},$$

which by derivating with respect to $\tilde{s}_1$ gives:

$$h''(\sigma_1) = \frac{d^2\sigma_2}{d\sigma_1^2} = \frac{\gamma(\tilde{s}_2)}{\gamma(\tilde{s}_1)^2}[\gamma'(\tilde{s}_2) - \gamma'(\tilde{s}_1)] = \frac{\gamma(\tilde{s}_2)}{\gamma(\tilde{s}_1)^2}\gamma''(\theta_{\tilde{s_1},\tilde{s}_2})(\tilde{s}_2 - \tilde{s}_1) \geq 0,$$

where $\theta_{\tilde{s_1},\tilde{s}_2} \in [\tilde{s}_2, \tilde{s}_1]$, and the inequality above is a consequence of the concavity of γ, and the invariance of $\mathcal{D}$. ☐

The next corollary follows directly from Lemma 2.2 and Proposition 2.1, and we will omit its proof.

Corollary 2.1. *Let $\overline{s} \in int\ \mathcal{D}$ and fix the control $u = (1, 1)$. Let $\tilde{t}$ be given as in Lemma 2.2.*
(i) If $\tilde{s}_1(\tilde{t}) = \tilde{s}_2(\tilde{t})$, we define a parametrized curve p as follows:

$$\begin{array}{rccl} p: & [-\tilde{t}, +\infty[& \rightarrow & \mathcal{D} \\ & t & \mapsto & \begin{cases} s(t, \overline{s}, (1, 1)), & \text{if } t \in [-\tilde{t}, 0[, \\ s(-t, \overline{s}, (0, 0)), & \text{if } t \in [0, \infty[. \end{cases} \end{array} \tag{2.11}$$

(ii) If $\tilde{s}_2(\tilde{t}) = 0$,we define a parametrized curve p as follows:

$$\begin{array}{rccl} p: & [-(\tilde{t} + s_1(\tilde{t}), +\infty[& \rightarrow & \mathcal{D} \\ & t & \mapsto & \begin{cases} (t + (\tilde{t} + s_1(\tilde{t})), 0), & \text{if } t \in [-(\tilde{t} + \tilde{s}_1(\tilde{t}), -\tilde{t}[, \\ s(t, \overline{s}, (1, 1)), & \text{if } t \in [-\tilde{t}, 0[, \\ s(-t, \overline{s}, (0, 0)), & \text{if } t \in [0, \infty[. \end{cases} \end{array} \tag{2.12}$$

Then, in both cases (i) and (ii), $p(\cdot)$ has its first component increasing.

We are now in position to define the subset $\mathcal{C}(\overline{s})$ mentioned above (see Fig. 1).

Definition 2.1. If $\overline{s} \in \text{int } \mathcal{D}$ and $p(\cdot)$ is given by Corollary 2.1, then we set $\mathcal{C}(\overline{s}) := p_{\geq}$.

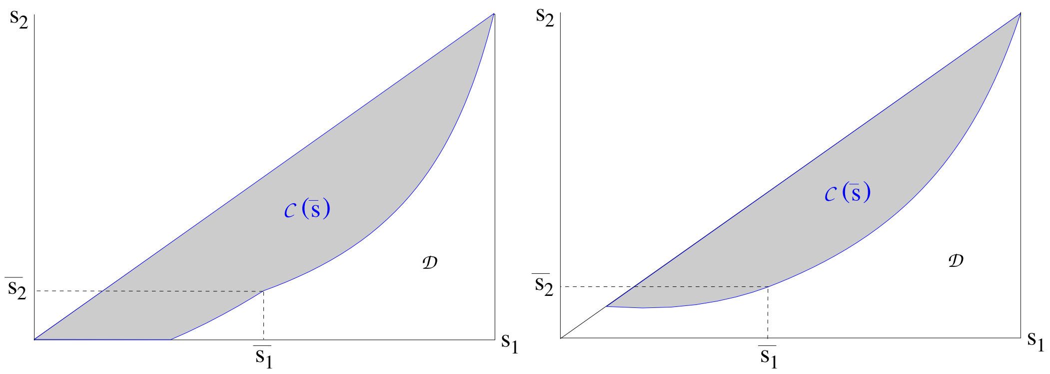

Figure 1: Plot of the set $\mathcal{C}(\overline{s})$ according to Definition 2.1. The boundary of $\mathcal{C}(\overline{s})$ is described by Lemma 2.2 and Proposition 2.1. *Picture left* : the backward curve starting at $\overline{s} = (1, 0.3)$ with $u_1 = u_2 = 1$ intersects the axis $x_2 = 0$, see Lemma 2.2. *Picture right*: the backward curve starting at $\overline{s} = (0.9, 0.32)$ with $u_1 = u_2 = 1$ intersects the first bisectrix.

2.3 Controllability in $\mathcal{C}(\overline{s})$

In this subsection we show that the target point $\overline{s}$ can be reached by any initial condition in $\mathcal{C}(\overline{s})$ by a control either with the first component constant equal to 1 or with the second one constant equal to zero. To do so, we will show that $\mathcal{C}(\overline{s})$ is equal to the set of backward time solutions starting from $\overline{s}$ and either with control first component equal to one or second component equal to zero.

First, recall that the system (1.3) is said *locally controllable* in time T at $s_0 \in \mathcal{D}$ if and only if for every neighborhood V of s_0, there exists a neighborhood W of s_0 such that for any pair $(y, z) \in W$, there exists an admissible control u steering y to z inside V in time T (see e.g. [29]).

We introduce the following admissible control set:

$$\mathcal{V} := \{v : [0, +\infty) \to [0, 1] \mid v \text{ meas.}\}.$$

Now, for $v \in \mathcal{V}$, $\overline{s} \in \mathcal{D}$, we call t_{u_1} (resp. t'_{u_2}) the exit time of $\mathcal{D}$ of the trajectory $t \longmapsto s(-t, \overline{s}, (u_1, 0))$ (resp. $t \longmapsto s(-t, \overline{s}, (1, u_2))$). We define the set $\mathcal{S}(\overline{s})$ by:

$$\mathcal{S}(\overline{s}) := \left(\bigcup_{u_1(\cdot) \in \mathcal{V}, t \in [0, t_{u_1})} s(-t, \overline{s}, (u_1, 0)) \right) \bigcup \left(\bigcup_{u_2 \in \mathcal{V}, t \in [0, t'_{u_2})} s(-t, \overline{s}, (1, u_2)) \right). \tag{2.13}$$

The next result show that $\mathcal{C}(\overline{s}) = \mathcal{S}(\overline{s})$ and is fundamental in order to prove the optimality result in $\mathcal{C}(\overline{s})$.

Proposition 2.2. *Let $\overline{s} = (\overline{s}_1, \overline{s}_2) \in$ int $\mathcal{D}$. Then we have $\mathcal{C}(\overline{s}) = \mathcal{S}(\overline{s})$.*

Proof. Let us first prove that $\mathcal{C}(\overline{s}) \subset \mathcal{S}(\overline{s})$. Take $s_0 \in \mathcal{C}(\overline{s})$, and consider the trajectory $(s_1(\cdot), s_2(\cdot)) := s(\cdot, s_0, (1, 0))$. If there exists $t_0 \geq 0$ such that $s(t_0) = \overline{s}$, then we have $s_0 = s(-t_0, \overline{s}, (1, 0)) \in \mathcal{S}(\overline{s})$ as was to be proved. Now, we suppose that for any $t \geq 0$, we have $s(t) \neq \overline{s}$ so that there exists $t_0 > 0$ such that $s(t_0) \in \partial\mathcal{D}$. Combining the inequality $0 < s_2(0) < s_1(0)$ and the fact that s_2 is decreasing, we necessarily have two cases: either the trajectory $(s_1(\cdot), s_2(\cdot))$ intersects at time t_0 the curve $t \longmapsto s(-t, \overline{s}, (1, 1))$ (case a), either it intersects at time t_0 the curve $t \longmapsto s(-t, \overline{s}, (0, 0))$ (case b).
Case a. Let $t_1 > 0$ be such that $\overline{s} = s(t_1, s(t_0), (1, 1))$. Then, let us define a control $u : [0, t_0 + t_1] \to U$ by $u = (1, 1)$ on $[0, t_1]$ and $u = (1, 0)$ on $[t_1 + t_0]$. By construction, we have $s(-t_1 - t_0, \overline{s}, u) = s_0$, and u is of the form $u = (1, u_2)$ with $u_2 \in \mathcal{V}$, and the result follows.
Case b. Let $t'_1 > 0$ be such that $\overline{s} = s(t'_1, s(t_0), (0, 0))$. Then, let us define a control $\tilde{u} : [0, t_0 + t'_1] \to U$ by

$\tilde{u} = (0,0)$ on $[0, t_1']$ and $\tilde{u} = (1,0)$ on $[t_1' + t_0]$. By construction, we have $s(-t_1' - t_0, \overline{s}, \tilde{u}) = s_0$, and $\tilde{u}$ is of the form $\tilde{u} = (u_1, 0)$ with $u_1 \in \mathcal{V}$, and the result follows.

Let us now prove that $\mathcal{S}(\overline{s}) \subset \mathcal{C}(\overline{s})$.

Take $s_0 \in \mathcal{S}(\overline{s})$, and assume first that $s_0 = s(-t_0, \overline{s}, (1, u_2))$ with $u_2 \in \mathcal{V}$ and $t_0 \in [0, t'_{u_2})$. For $t \in [0, t'_{u_2})$, set $\tilde{s}(t) := s(-t, \overline{s}, (1, u_2))$, and $\hat{s}(t) := s(-t, \overline{s}, (1,1))$. Similarly as in the proof of Lemma 2.1, one can prove that $\tilde{s}_1(t) = \hat{s}_1(t)$ and $\tilde{s}_2(t) \geq \hat{s}_2(t)$ for all $t \in [0, t'_{u_2})$.
Case a. If the boundary of $\mathcal{C}(\overline{s})$ is given by (2.11), then we necessarily have $\tilde{s}_1(t'_{u_2}) = \tilde{s}_2(t'_{u_2})$. Now, we have $\tilde{s}_1(t_0) = \hat{s}_1(t_0)$, $\tilde{s}_2(t_0) \geq \hat{s}_2(t_0)$, $\hat{s}(t_0) \in \partial\mathcal{C}(\overline{s})$, and as $t_0 \leq t'_{u_2}$, we get that $\tilde{s}(t_0) \in \mathcal{C}(\overline{s})$.
Case b. Assume that the boundary of $\mathcal{C}(\overline{s})$ is given by (2.12). If we have $\tilde{s}_1(t'_{u_2}) = \tilde{s}_2(t'_{u_2})$, we conclude by case a. Now, assume that $\tilde{s}_2(t'_{u_2}) = 0$. Then, we have $\tilde{s}_1(t_0) = \hat{s}_1(t_0)$, $\tilde{s}_2(t_0) \geq \max(0, \hat{s}_2(t_0))$, $(\hat{s}_1(t_0), \max(0, \hat{s}_2(t_0))) \in \partial\mathcal{C}(\overline{s})$ so that $t_0 \leq t'_{u_2}$ implies $\tilde{s}(t_0) \in \mathcal{C}(\overline{s})$.

Suppose now that $s_0 = s(-t_0, \overline{s}, (u_1, 0))$ with $u_1 \in \mathcal{V}$ and $t_0 \in [0, t_{u_1})$. For $t \in [0, t_{u_1})$, set $\tilde{s}(t) := s(-t, \overline{s}, (u_1, 0))$, $\check{s}(t) := s(-t, \overline{s}, (0,0))$, and $\hat{s}(t) := s(-t, \overline{s}, (1,0))$ Similarly as in the previous case, we obtain:

$$\tilde{s}_2(t) = \check{s}_2(t) = \hat{s}_2(t), \quad \text{and} \quad \hat{s}_1(t) \leq \tilde{s}_1(t) \leq \check{s}_1(t), \quad \forall t \in [0, t_{u_1}). \tag{2.14}$$

Notice that $\overline{s}_2 > 0$ so that at the exit time t_{u_1}, we necessarily have $\tilde{s}_2(t_{u_1}) = \tilde{s}_1(t_{u_1})$. Moreover, combining the fact that $(\check{s}_1(t_0), \check{s}_2(t_0)) \in \partial\mathcal{C}(\overline{s})$ and that $(\hat{s}_1(t), \hat{s}_2(t)) \in \mathcal{C}(\overline{s})$ for all $t \in [0, t_{u_1})$, we obtain that $\tilde{s}(t_0) \in \mathcal{C}(\overline{s})$ as was to be proved. □

2.4 Controllability outside of $\mathcal{C}(\overline{s})$

For future reference, we define $f, g : \mathbb{R}_+^2 \to \mathbb{R}^2$ by

$$f(s) := \begin{pmatrix} -\mu(s_1)(s_{in} - s_1) \\ -\mu(s_2)(s_{in} - s_2) \end{pmatrix}, \; g(s) := \begin{pmatrix} s_{in} - s_1 \\ s_1 - s_2 \end{pmatrix}, \; s = (s_1, s_2).$$

Also, let $\gamma(s) := \mu(s)(s_{in} - s)$. Now, let $\Delta_0 \subset \; Int\mathcal{D}$ the set of points where f and g are collinear :

$$\begin{aligned} \Delta_0 :&= \{(s_1, s_2) \in \; Int\mathcal{D} \mid \; \det(f(s), g(s)) = 0\} \\ &= \{(s_1, s_2) \in \; Int\mathcal{D} \mid \mu(s_1)(s_1 - s_2) - \mu(s_2)(s_{in} - s_2) = 0\}. \end{aligned} \tag{2.15}$$

and we define Δ_0^+ (resp. Δ_0^-) as the set points of $\mathcal{D}$ such that $\det(f(s), g(s)) > 0$ (resp. $\det(f(s), g(s)) < 0$). We now discuss the controllability problem for initial condition outside of $\mathcal{C}(\overline{s})$. From what we saw in the previous subsection, given a initial condition $s_0 \in \mathcal{D} \setminus \mathcal{C}(\overline{s})$, one way to reach $\overline{s}$ is to reach the set $\mathcal{C}(\overline{s})$. Obviously, this relaxed problem highly depends on $\overline{s}$. Hereafter, we introduce the vector $e_z = (0,0,1) \in \mathbb{R}^3$ and orthogonal to the plane (s_1, s_2).

Lemma 2.3. *The set Δ_0 satisfies the following properties.*
(i) For any $s \in \mathcal{D}$, we have $((f+g)(s) \wedge f(s)) \cdot e_z \geq 0 \iff s \in \Delta_0^-$.
(ii) There exist a continuous mapping $s_1 \in [0, s_{in}] \longmapsto \zeta(s_1)$ such that Δ_0 coincides with the graph of ζ. Moreover, we have $\zeta(0) = 0$, $\zeta(s_{in}) = s_{in}$, and ζ is increasing on $(0, s_{in})$.

Proof. The proof of (i) follows from a direct computation. To prove (ii), let us consider the C^1-mapping $\rho : (s_1, s_2) \in \mathcal{D} \longmapsto \rho(s_1, s_2) := -\mu(s_1)(s_1 - s_2) + \mu(s_2)(s_{in} - s_2)$ so that $\rho(s_1, s_2) = 0$ iff $s \in \Delta_0$. For $(s_1, s_2) \in \mathcal{D} \backslash \{(0,0), (s_{in}, s_{in})\}$, we have $\frac{\partial \rho}{\partial s_2}(s_1, s_2) = \mu(s_1) - \mu(s_2) + \mu'(s_2)(s_{in} - s_2) > 0$. Hence, we are in position to apply the implicit function Theorem to ρ. For any $s_1^0 \in (0, s_{in})$, there exists a neighborhood $\mathcal{W}$ of s_1^0, a function $\zeta : \mathcal{W} \to \mathbb{R}$ of class C^1 such that $s_1 \in \mathcal{W}$, $(s_1, s_2) \in \mathcal{D}$ together with $\rho(s_1, s_2) = 0$ if and only if $s_2 = \zeta(s_1)$. For $s_1 \in \mathcal{W}$, the derivative of ζ is given by:

$$\zeta'(s_1) = \frac{ds_2}{ds_1} = \frac{\mu'(s_1)(s_1 - s_2) + \mu(s_1)}{\mu(s_1) - \mu(s_2) + \mu'(s_2)(s_{in} - s_2)} > 0. \tag{2.16}$$

This proves that ζ is increasing over $(0, s_{in})$. Moreover, as $\mu(s_1) - \mu(s_2) + \mu'(s_2)(s_{in} - s_2)$ can be zero only if $s_1 = 0$ or $s_1 = s_{in}$, this proves that ζ is defined over $(0, s_{in})$ and that $s_1 \longmapsto s_2 = \zeta(s_1)$ satisfies (2.16) over $(0, s_{in})$. By letting $\zeta(0) = 0$ and $\zeta(s_{in}) = s_{in}$, we can extend ζ continuously on $[0, s_{in}]$. □

Condition (i) of Lemma 2.3 is important for the controllability of (1.3). From a geometrical point of view, given a point s_0 below Δ_0 (resp. above Δ_0), this condition means that locally it is possible to increase (resp. decrease) s_2 by taking a control $u = 1$ and $u = 0$.

Proposition 2.3. *Let $s(\cdot) = (s_1(\cdot), s_2(\cdot)) := s(\cdot, (0,0), (1,1))$. Then, the following properties are satisfied:*
(i) The trajectory $s(\cdot)$ is defined over $[0,+\infty[$ and $s_i(\cdot)$ is increasing on $[0,+\infty[$ for each $i \in \{1,2\}$.
(ii) When t goes to infinity, we have:

$$\lim_{t\to+\infty} s(t) = (s_{in}, s_{in}), \quad \lim_{t\to+\infty} \frac{s_{in} - s_2(t)}{s_{in} - s_1(t)} = +\infty. \tag{2.17}$$

Proof. Let $[0, t_f)$ be the maximal time interval where s is defined. As $s_1(0) < s_{\text{in}}$, one has $s_1(t) < s_{\text{in}}$ for all $t \in [0, t_f[$, so $\dot{s_1} > 0$ on $[0, t_f[$, hence $s_1(\cdot)$ is increasing on $[0, t_f[$. Notice that at time 0, we have $\dot{s}_1(0) > 0$, $\dot{s}_2(0) = 0$ and that:

$$\ddot{s}_2 = -[\mu'(s_2)(s_{in} - s_2)\dot{s}_2 - \mu(s_2)\dot{s}_2] + \dot{s}_1 - \dot{s}_2,$$

hence we obtain that $\ddot{s}_2 > 0$ whenever $\dot{s}_2 = 0$ (using that $\dot{s_1} > 0$ for all time $t \geq 0$), which proves that $\dot{s}_2 \geq 0$. Consequently, as for each $i \in \{1,2\}$, $s_i(\cdot)$ is upper bounded by s_{in}, one has that $s_i(\cdot)$ is defined on $[0,+\infty[$ and converges to some s_i^∞ when t tends to $+\infty$. We necessarily have $s_i^\infty = s_{\text{in}}$ by monotonicity.

Let us now define z by $z := \frac{s_{in}-s_2}{s_{in}-s_1}$. We can check that z satisfies:

$$\dot{z} = -1 + z[\mu(s_1) - \mu(s_2)].$$

If z is bounded on $[0,+\infty)$, then we have $\dot{z}(t) \to -1$ when t goes to infinity which contradicts the boundedness of z. Hence, z is unbounded on $[0,+\infty)$. Now, one can check that the linearized of (1.3) with $u_1 = u_2 = 1$ around the equilibrium (s_{in}, s_{in}) has a double eigenvalue $\lambda = \mu(s_{in}) - 1 < 0$ and that it is a stable improper node with eigenvector $(0,1)$. So, each trajectory of the linearized system that converges to (s_{in}, s_{in}) is tangent to the separatrice $(1,0)$. By applying Hartmann-Grobmann's Theorem (see e.g. [25]) on (1.3) with $u_1 = u_2 = 1$ around the equilibrium (s_{in}, s_{in}), we conclude that z converges to infinity when t goes to infinity. □

The proposition above ensures the existence of a continuous mapping $F : [0, s_{in}] \to [0, s_{in}]$ such that $s_2 = F(s_1)$, $(s_1, s_2) \in [0, s_{in}] \times [0, s_{in}]$ if and only if there exists $t \geq 0$ satisfying $(s_1, s_2) = s(t, (0,0), (1,1))$. From the regularity property of $s(\cdot, (0,0), (1,1))$, we can argue that F is of class C^1 over $[0, s_{in})$. We call $\mathcal{R}$ the graph of F and $\mathcal{R}_-$ (resp. $\mathcal{R}_+$) the set of points of $\mathcal{D}$ below the $\mathcal{R}$ (resp. above $\mathcal{R}$). We can prove the following Lemma.

Lemma 2.4. *There exists a neighborhood in $\mathcal{D}$ of $(0,0)$ in which we have $h > F$.*

Proof. Recall that both curves contain the origin. Also by derivating ρ with respect to s_1, we can check that $h'(0) = 0$. For any pair (s_1, s_2) such that $s_2 = F(s_1)$, we have:

$$F'(s_1) = \frac{\mu(s)_2(s_{in} - s_2) - (s_1 - s_2)}{(\mu(s_1) - 1)(s_{in} - s_1)},$$

which by derivating with respect to s_1 gives for $s_1 = s_2 = 0$: $F''(0) = \frac{1}{s_{in}}$. By derivating two times ρ, we find for $s_1 = s_2 = 0$ that $h''(0) = \frac{2}{s_{in}} > F''(0)$. We then conclude using $F'(0) = 0$. □

The proof of the next proposition is illustrated in Fig. 2.

Proposition 2.4. *(i) Assume that $\overline{s} \in \Delta_0^-$. Then, for any initial condition $s_0 \in \mathcal{D}$, there exists an admissible control $u \in \mathcal{U}$ steering s_0 to $\overline{s}$.*
(ii) Assume that $\overline{s} \in \Delta_0^+ \cap \mathcal{R}_-$. Then, for any initial condition in $s_0 \in \mathcal{D}$, there exists an admissible control $u \in \mathcal{U}$ steering s_0 to $\overline{s}$.

Proof. Let us prove (i). We only study the case where $s_0 \notin \mathcal{C}(\overline{s})$. First assume that $s_0 := (s_1^0, s_2^0)$ is such that $s_1^0 = \overline{s}_1$ and $s_2^0 < \overline{s}_2$. There exists a control $\overline{u}_1$ such that $\mu(\overline{s}_1) = \overline{u}_1$ implying $\dot{s}_1 = 0$. As $\overline{s} \in \Delta_0^-$ and from Lemma 2.3, we have that for each $s_2 \in [s_2^0, \overline{s}_2]$, then $s_2 \in \Delta_0^-$. Hence, we have:

$$-\mu(\overline{s}_1)(\overline{s}_1 - s_2) + \mu(s_2)(s_{in} - s_2) < 0$$

It follows that by taking $u_2 := \overline{u}_1$, we have $\dot{s}_2 = -\mu(s_2)(s_{in} - s_2) + \overline{u}_1(\overline{s}_1 - s_2) > 0$, and the target $\overline{s}$ is reached in finite time as $\overline{s}_2$ is not an equilibrium of $\dot{s}_2 = 0$ with $u_2 = \overline{u}_1$. Now, if $s_1^0 > \overline{s}_1$ (resp. $s_1^0 > \overline{s}_1$), we apply the control $(0,0)$ (resp. $(1,1)$) in order to reach $s = \overline{s}_1$ with $s_2 < \overline{s}_2$, and we can apply the procedure described above in order to reach the target in finite time.

Let us prove (ii). As $\overline{s} \in \mathcal{R}_-$, there exists $t_0 > 0$ such that the mapping $t \longmapsto \tilde{s}(t) := s(-t, \overline{s}, (1,1))$ satisfies $\tilde{s}_2(t_0) = 0$ and $\tilde{s}_1(t_0) > 0$. Hence $\mathcal{C}(\overline{s})$ contains the set

$$E := \{\sigma = (\sigma_1, \sigma_2) \in \mathcal{D} \mid 0 \leq \sigma_2 \leq \sigma_1, \ \sigma_1 \leq \tilde{s}_1(t_0)\}.$$

Let us be given $s_0 \in \mathcal{D} \backslash \mathcal{C}(\overline{s})$, and consider the mapping $t \longmapsto s(t) := s(t, s_0, (0,0))$. We have that both s_i, $i = 1, 2$ are positive, decreasing and converge to zero. Hence, there exists $t > 0$ such that $s(t) \in E$. We conclude from Proposition 2.2. $\square$

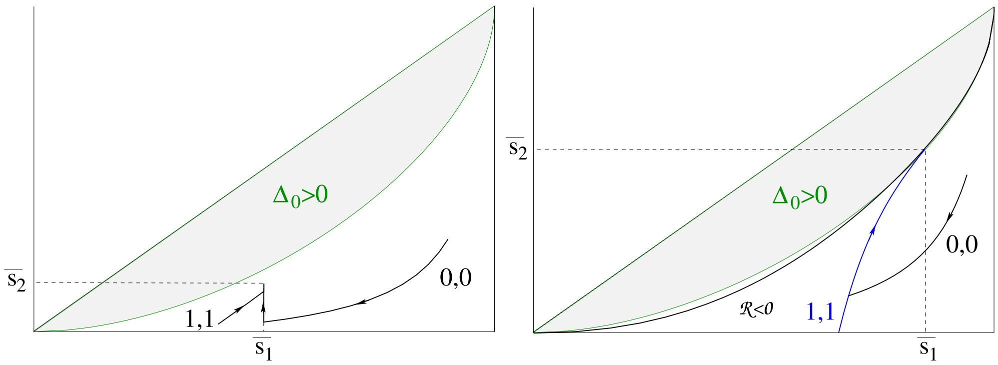

Figure 2: *Picture left* : in red the curve $\mathcal{R}$ and in black the trajectory described in the proof of Proposition 2.4 (i) in order to reach the target $\overline{s} = (1, 0.3)$. *Picture right*: in red the curve $\mathcal{R}$ and in black the set Δ_0 with $\overline{s} = (1.7, 1.127)$. The two others curves illustrate the proof of Proposition 2.4 (ii) in order to reach the target.

Lemma 2.4 implies that $\Delta_0^+ \cap \mathcal{R}_-$ is empty near the origin, nevertheless we can check numerically that this set is non-empty for some values of s, see Fig. 2.

Remark 2.3. *(i) From a practical point of view, the target point is often chosen in Δ_0^- in order to guarantee the existence of simple control laws driving any initial condition in $\mathcal{D}$ to the target (see the proof of Proposition 2.4 and 2).*
(ii) In the case where the target $\overline{s}$ is in Δ_0^+ and above $\mathcal{R}$, the target can be non-locally controllable for any time $T > 0$ in the sense given above. Indeed, take $s_0 \in \mathcal{D}$ sufficiently close to $\overline{s}$, and suppose that $s_2^0 < \overline{s}_2$ and $s_1^0 = \overline{s}_1$. Then, condition $((f(s) + g(s)) \wedge g(s)) \cdot e_z < 0$ implies that the forward solution of (1.3) *starting from s_0 with either $u = 0$ or $u = 1$ cannot intersect the backward solution of* (1.3) *starting from $\overline{s}$ with $u = 1$ and $u = 0$.*
(iii) For brevity and practical reasons, we will not investigate the case where the target is in Δ_0^+.

3 Optimality result in $\mathcal{C}(\overline{s})$

In this section, we give explicitly the value function of the minimum time problem on the set $\mathcal{C}(\overline{s})$ for $\overline{s} \in \text{int } \mathcal{D}$. Our aim is to reach a given target point $\overline{s} = (\overline{s_1}, \overline{s_2}) \in \text{int } \mathcal{D}$ from an initial condition $s_0 \in \mathcal{C}(\overline{s})$ in a minimal amount time. Let us introduce some notations. We will denote by $\mathcal{U}(s_0)$ the set of control that allows to reach $\overline{s}$ from s_0, i.e.,

$$\mathcal{U}(s_0) := \{u \in \mathcal{U} \mid \ \exists\, t_u(s_0) \in [0, +\infty[\text{ s.t. } s(t_u(s_0), s_0, u) = \overline{s}\}.$$

From Proposition 2.2, one has $\mathcal{U}(s_0) \neq \emptyset$, for all $s_0 \in \mathcal{C}(\overline{s})$. We want to minimize the following cost function with respect to $u \in \mathcal{U}$:

$$J(s_0, u) := \begin{cases} \int_0^{t_u(s_0)} dt, & \text{if } u \in \mathcal{U}(s_0), \\ +\infty, & \text{else.} \end{cases} \tag{3.1}$$

For $s \in \mathcal{C}(\overline{s})$, the value function is defined by $v(s) = \inf_{u \in \mathcal{U}} J(s,u)$, and it represents the minimal time to reach $\overline{s}$ from s.

Proposition 3.1. *For any $s \in \mathcal{C}(\overline{s})$, the value function satisfies:*

$$v(s) = \max\left\{\int_{s_1}^{\overline{s_1}} \frac{d\sigma}{(1-\mu(\sigma))(s_{in}-\sigma)}, \int_{s_2}^{\overline{s_2}} \frac{d\sigma}{-\mu(\sigma)(s_{in}-\sigma)}\right\}. \tag{3.2}$$

Proof. Let $s_0 = (s_1^0, s_2^0) \in \mathcal{C}(\overline{s})$. We know from Proposition 2.2 that there exists a control $u \in \mathcal{U}$ of the form $(1, u_2)$ or $(u_1, 0)$ and a time $t_u \in [0, +\infty[$ such that

$$s(t_u, s_0, u) = \overline{s}. \tag{3.3}$$

First, suppose that u is of the form $(1, u_2)$ and set $(s_1(\cdot), s_2(\cdot)) := s(\cdot, s_0, u)$. Therefore one has

$$\dot{s_1} = (1-\mu(s_1))(s_{\text{in}} - s_1), \tag{3.4}$$

hence s_1 is increasing and:

$$s_1(t) < \overline{s}_1, \quad \forall t \in [0, t_u). \tag{3.5}$$

We show that $t_u = v(s_0)$. It is obvious that $v(s_0) \leq t_u$ by definition of v, so let us suppose $v(s_0) < t_u$. Thus, there exists $u^* = (u_1^*, u_2^*) \in \mathcal{U}$ such that $s(v(s_0), s_0, u^*) = \overline{s}$. Set $(s_1^*(\cdot), s_2^*(\cdot)) = s(\cdot, s_0, u^*)$. As $u_1^* \leq 1$ and $s_1^*(0) = s_1(0) = s_1^0$, we obtain by (1.3):

$$s_1^*(t) \leq s_1(t), \quad \forall t \in [0, +\infty). \tag{3.6}$$

Therefore, combining (3.5) and (3.6), we get:

$$\overline{s}_1 = s_1^*(v(s_0)) \leq s_1(v(s_0)) < \overline{s}_1, \tag{3.7}$$

which is a contradiction, hence $t_u = v(s_0)$. Let us now calculate t_u. One has:

$$t_u = \int_0^{t_u} dt = \int_0^{t_u} \frac{\dot{s_1}(t)}{(1-\mu(s_1(t)))(s_{\text{in}} - s_1(t))} dt = \int_{s_1^0}^{\overline{s}_1} \frac{d\sigma}{(1-\mu(\sigma))(s_{\text{in}} - \sigma)}. \tag{3.8}$$

Finally let us show that

$$\int_{s_2^0}^{\overline{s}_2} \frac{d\sigma}{-\mu(\sigma)(s_{\text{in}} - \sigma)} \leq \int_{s_1^0}^{\overline{s}_1} \frac{d\sigma}{(1-\mu(\sigma))(s_{\text{in}} - \sigma)}. \tag{3.9}$$

Remark that $\frac{1}{-\mu(s)(s_{\text{in}}-s)} < 0$, for all $s \in]0, s_{\text{in}}[$. Therefore if $s_2^0 \leq \overline{s}_2$, one has $\int_{s_2^0}^{\overline{s}_2} \frac{d\sigma}{-\mu(\sigma)(s_{\text{in}}-\sigma)} \leq 0$ and the result is obvious. Let us now suppose that $s_2^0 > \overline{s}_2$ and set φ as the solution of the Cauchy problem:

$$\begin{cases} \dot{\varphi} = -\mu(\varphi)(s_{\text{in}} - \varphi), \\ \varphi(0) = s_2^0. \end{cases}$$

Then, φ is decreasing and converges to zero when t tends to $+\infty$. Thus, there exists $t_0 \in [0, +\infty)$ such that $\varphi(t_0) = \overline{s}_2$. Therefore, one has

$$\int_{s_2^0}^{\overline{s}_2} \frac{d\sigma}{-\mu(\sigma)(s_{\text{in}} - \sigma)} = \int_{\varphi(0)}^{\varphi(t_0)} \frac{d\sigma}{-\mu(\sigma)(s_{\text{in}} - \sigma)} = \int_0^{t_0} dt = t_0. \tag{3.10}$$

Moreover, as $u_2 \geq 0$ and $s_2(0) = \varphi(0)$ one has by (1.3) that $\varphi(t) \leq s_2(t)$ for all $t \in [0, +\infty[$. Consequently, as φ is decreasing with $\varphi(t_0) = s_2(t_u) \geq \varphi(t_u)$, one has $t_0 \leq t_u$, which by (3.8) and (3.10) gives (3.9) as wanted and proves (3.2).

Now, we investigate the case where u is of the form $(u_1, 0)$. We set $(s_1(\cdot), s_2(\cdot)) = s(\cdot, s_0, u)$. We have that s_2 is decreasing and therefore $s_2^0 > \overline{s}_2$. Similarly as in the previous case, we can show that $v(s_0) = t_u$ (using the fact that $s_2^0 > \overline{s}_2$), and that $t_u = \int_{s_2^0}^{\overline{s}_2} \frac{d\sigma}{-\mu(\sigma)(\sin - \sigma)}$. Finally, we show that:

$$\int_{s_1^0}^{\overline{s}_1} \frac{d\sigma}{(1-\mu(\sigma))(s_{\text{in}} - \sigma)} \leq \int_{s_2^0}^{\overline{s}_2} \frac{d\sigma}{-\mu(\sigma)(s_{\text{in}} - \sigma)}. \tag{3.11}$$

First, if $s_1^0 \geq \overline{s}_1$, then $\int_{s_1^0}^{\overline{s}_1} \frac{d\sigma}{(1-\mu(\sigma))(s_{\rm in}-\sigma)} \leq 0$ and (3.11) is obvious. If now $s_1^0 < \overline{s}_1$, we consider φ as the solution of the following Cauchy problem:

$$\begin{cases} \dot{\varphi} = (1-\mu(\varphi))(s_{\rm in} - \varphi), \\ \varphi(0) = s_1^0. \end{cases}$$

As φ is increasing, there exists $t_0 > 0$ such that $\int_{s_1^0}^{\overline{s}_1} \frac{d\sigma}{(1-\mu(\sigma))(s_{\rm in}-\sigma)} = t_0$ with $\varphi(t_0) = \overline{s}_1$. Moreover, as φ satisfies the same ODE as s_1 with the constant control equal to 1 in place of u_1, we obtain $\varphi(t_0) = s_1(t_u) = \overline{s}_1 \leq \varphi(t_u)$, so that $t_0 \leq t_u$ (as φ is increasing). This concludes the proof. $\square$

Let $\tilde{t}$ is the exit time of $\mathcal{D}$ of the parametrized curve $t \longmapsto s(-t, \overline{s}, (1,0))$ We define $\Gamma \subset \mathcal{C}(\overline{s})$ by:

$$\Gamma = \{\sigma \in \mathcal{D} \mid \exists t \in [0, \tilde{t}] \ \sigma = s(-t, \overline{s}, (1,0))\}.$$

The previous proposition implies characterization of optimal controls in $\mathcal{C}(\overline{s})$ (see Fig. 3):

- If $s_0 \in \mathcal{C}(\overline{s})$ is above Γ, there exist infinitely many controls of the form $(u_1, 0)$, $u_1 \in \mathcal{V}$, steering s_0 to $\overline{s}$.
- If $s_0 \in \mathcal{C}(\overline{s})$ is below Γ, there exist infinitely many controls of the form $(1, u_2)$, $u_2 \in \mathcal{V}$, steering s_0 to $\overline{s}$.

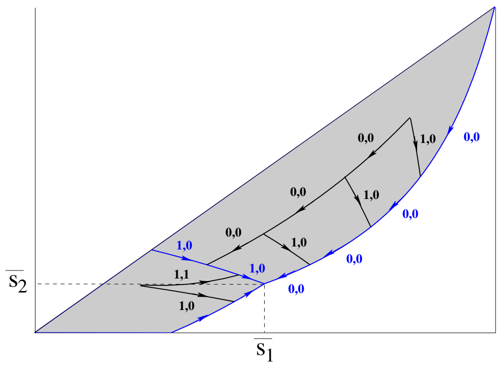

Figure 3: In the case where an initial condition s_0 is in $\mathcal{C}(\overline{s})$ with $\overline{s} = (1, 0.3)$, there exist infinitely many trajectories steering s_0 to $\overline{s}$ in time $v(s_0)$, where v is given by (3.2). When s_0 is below the curve Γ, the control u_1 is always equal to 1 whereas if s_0 is above Γ, then the control u_2 is always equal to 0.

4 Optimality result outside of $\mathcal{C}(\overline{s})$

In this section, we study the minimal time problem (1.6) whenever initial conditions are not in $\mathcal{C}(\overline{s})$. In view of (1.2), the target may be non-locally controllable (see Remark 2.3) for instance, if $\overline{s} \in \Delta_0^+ \cap \mathcal{R}^+$. In order to overcome this difficulty, we will mainly study the case where $\overline{s} \in \Delta_0^-$ (see Proposition 2.4).

For future reference, we define a set Δ_{SA} by:

$$\Delta_{SA} := \{(s_1, s_2) \in \mathcal{D} \mid \ \det(g(s), [f,g](s)) = 0\}, \tag{4.1}$$

where $[f, g]$ denotes the Lie bracket of f and g, see e.g. [16]. Similarly, let Δ_{SA}^+ (resp. Δ_{SA}^-) the set points of $\mathcal{D}$ such that $\det(g(s), [f, g](s)) > 0$ (resp. $\det(g(s), [f, g](s)) < 0$). In the rest of this section, we always consider initial conditions $s_0 \in \mathcal{D} \backslash \mathcal{C}(\overline{s})$. Recall that $\gamma(s) := \mu(s)(s_{in} - s)$.

4.1 Pontryagin maximum principle

In this part, we apply Pontryagin maximum principle (PMP) in order to derive necessary optimality conditions on problem (1.6), see [26]. The Hamiltonian $H = H(s_1, s_2, \lambda_0, \lambda_1, \lambda_2, u_1, u_2)$ associated to (1.3) is defined by:

$$H := -\lambda_1 \mu(s_1)(s_{in} - s_1) - \lambda_2 \mu(s_2)(s_{in} - s_2) + \lambda_0 + \lambda_1 (s_{in} - s_1) u_1 + \lambda_2 (s_1 - s_2) u_2. \tag{4.2}$$

Pontryagin maximum principle can be stated as follows. Let $u := (u_1, u_2)$ an optimal control and $s := (s_1, s_2)$ the associated trajectory. There exists $t_f > 0$, $\lambda_0 \leq 0$ and $\lambda : [0, t_f] \to \mathbb{R}^2$ satisfying the adjoint equations $\dot{\lambda} = -\frac{\partial H}{\partial s}$, that is:

$$\begin{cases} \dot{\lambda}_1 = \lambda_1[\mu'(s_1)(s_{in} - s_1) - \mu(s_1) + u_1] - \lambda_2 u_2 = \lambda_1[\gamma'(s_1) + u_1] - \lambda_2 u_2, \\ \dot{\lambda}_2 = \lambda_2[\mu'(s_2)(s_{in} - s_2) - \mu(s_2) + u_2] = \lambda_2[\gamma'(s_2) + u_2], \end{cases} \tag{4.3}$$

moreover, we have the maximization condition:

$$u(t) \in \mathrm{argmax}_{0 \leq \omega_2 \leq \omega_1 \leq 1} H(s_1(t), s_2(t), \lambda_0, \lambda_1(t), \lambda_2(t), \omega_1, \omega_2). \tag{4.4}$$

An *extremal trajectory* is a sextuplet $(s_1(\cdot), s_2(\cdot), \lambda_0, \lambda_1(\cdot), \lambda_2(\cdot), u(\cdot))$ satisfying (1.3)-(4.3)-(4.4). Next, we consider only *normal* extremal trajectories, that is we consider only extremal trajectories for which $\lambda_0 < 0$. Without any loss of generality, we assume that $\lambda_0 = -1$. The Hamiltonian is zero along an extremal trajectory (as t_f is free), thus we obtain:

$$-\lambda_1\mu(s_1)(s_{in} - s_1) - \lambda_2\mu(s_2)(s_{in} - s_2) - 1 + \lambda_1(s_{in} - s_1)u_1 + \lambda_2(s_1 - s_2)u_2 = 0. \tag{4.5}$$

Given the control constraints, we introduce the the two *switching functions* that will allow to determine an extremal control:

$$\begin{cases} \phi_1 := \lambda_1(s_{in} - s_1), \\ \phi_2 := \lambda_1(s_{in} - s_1) + \lambda_2(s_1 - s_2) = \phi_1 + \lambda_2(s_1 - s_2). \end{cases} \tag{4.6}$$

We say that t_0 is a *switching point* if for any neighborhood $\mathcal{W}$ of t_0, the control u is non-constant in $\mathcal{W}$. At a switching point, we necessarily have $\phi(t_0) = 0$. By differentiating with respect to the time, we find:

$$\begin{cases} \dot{\phi}_1 = \lambda_1\mu'(s_1)(s_{in} - s_1)^2 - \lambda_2(s_{in} - s_1)u_2, \\ \dot{\phi}_2 = \lambda_1\mu'(s_1)(s_{in} - s_1)^2 + \lambda_2[\gamma'(s_2)(s_1 - s_2) + \gamma(s_2) - \gamma(s_1) + (u_1 - u_2)(s_{in} - s_1)]. \end{cases} \tag{4.7}$$

Remark 4.1. *The control constraint set E implies the particular choice of the second switching function ϕ_2 in* (4.6)*. Notice that both controls are not independent in* (4.4)*, which justifies this choice.*

By using (4.4), we obtain the following characterization of an extremal control.

Proposition 4.1. *Let $u = (u_1, u_2)$ an extremal control defined on $[0, t_f]$. Then, we have for a.e. $t \in [0, t_f]$:*

- *(i) $\phi_1(t) > 0$, $\phi_2(t) > \phi_1(t) \Longrightarrow u_1(t) = u_2(t) = 1$ and $\lambda_2(t) > 0$*
- *(ii) $\phi_1(t) > 0$, $\phi_2(t) < \phi_1(t) \Longrightarrow u_1(t) = 1$, $u_2(t) = 0$ and $\lambda_2(t) < 0$.*
- *(iii) $\phi_1(t) > 0$, $\phi_2(t) = \phi_1(t) \Longrightarrow u_1(t) = 1$, $u_2(t) \in [0, 1]$, and $\lambda_2 \equiv 0$.*
- *(iv) $\phi_1(t) = 0$, $\phi_2(t) > 0 \Longrightarrow u_1(t) = u_2(t) = 1$.*
- *(v) $\phi_1(t) = 0$, $\phi_2(t) < 0 \Longrightarrow u_1(t) \in [0, 1]$, $u_2(t) = 0$.*
- *(vi) $\phi_1(t) < 0$, $\phi_2(t) > 0 \Longrightarrow u_1(t) = u_2(t) = 1$.*
- *(vii) $\phi_1(t) < 0$, $\phi_2(t) = 0 \Longrightarrow u_1(t) = u_2(t) \in [0, 1]$.*
- *(viii) $\phi_1(t) < 0$, $\phi_2(t) < 0 \Longrightarrow u_1(t) = u_2(t) = 0$.*

Proof. One can see that the mapping $(v_1, v_2) \longmapsto (v_1 + v_2, v_2)$ is a one-to-one correspondance between the set $F := \{v := (v_1, v_2) |\ v_1 \geq 0, v_2 \geq 0,\ v_1 + v_2 \leq 1\}$ and E. Therefore, maximizing (4.4) with respect to $\omega := (\omega_1, \omega_2) \in E$ is equivalent to maximize

$$f(v_1, v_2) := v_1\phi_1(t) + v_2\phi_2(t)$$

with respect to $v \in F$ for a.e. $t \in [0, t_f]$.
Let us now prove (i), (ii) and (iii). If $\phi_1(t) > 0$ and $\phi_2(t) > \phi_1(t)$, the maximum of f is achieved by taking $v_2(t) = 1$, thus $u_2(t) = 1$, $v_1(t) = 0$, and $u_1(t) = 1$. If $\phi_1(t) > 0$ and $\phi_2(t) < \phi_1(t)$, then the maximum of

f is achieved by taking $v_1(t) = 1$, hence $v_2(t) = 0$ and $u_1(t) = 1$, $u_2(t) = 0$. If $\phi_1(t) = \phi_2(t) > 0$, then the maximum of f is achieved by taking $v_1(t) + v_2(t) = 1$, that is $u_1(t) = 1$, thus $u_2(t) \in [0, 1]$. To conclude on the sign of $\lambda_2(t)$ for (i) and (ii), notice that we have $\lambda_2(s_1 - s_2) = \phi_2 - \phi_1$. For (iii), λ_2 is zero everywhere by Cauchy-Lipschitz Theorem (indeed one has $\phi_1(t) = \phi_2(t)$, which together with (4.3) implies that $\lambda_2 \equiv 0$).
Let us now prove (iv), (v). One cannot have $\phi_1(t) = \phi_2(t) = 0$ (otherwise we would have $\lambda_1(t) = \lambda_2(t) = 0$ in contradiction with (4.5)). If $\phi_1(t) = 0$ and $\phi_2(t) > 0$, the maximum of f is achieved by taking $v_2(t) = 1$, hence $u_2(t) = 1$ and $u_1(t) = 1$. If $\phi_1(t) = 0$ and $\phi_2(t) < 0$, then the maximum of f is achieved by taking $v_2(t) = 0$, hence $u_2(t) = 0$ and $u_1(t) = v_1(t) \in [0, 1]$.
Let us now prove (vi),(vii),(viii). When $\phi_1(t) < 0$ and $\phi_2(t) > 0$, the maximum of f is achieved by taking $v_2(t) = 1$, thus $v_1(t) = 0$ and $u_2(t) = u_1(t) = 1$, whereas if $\phi_1(t) < 0$ and $\phi_2(t) < 0$, the maximum of f is achieved by taking $v_1(t) = v_2(t) = u_1(t) = u_2(t) = 0$. When $\phi_1(t) < 0$ and $\phi_2(t) = 0$, we get $v_1(t) = 0$. It follows that $u_1(t) = v_2(t) = u_2(t) \in [0, 1]$. □

From (4.3), we have that λ_2 is always of constant sign, or constant equal to zero. Moreover, we have $\lambda_2 \equiv 0$ only in case (iii) of the previous proposition, which implies that u_1 is constant equal to 1. As $s^0 \notin \mathcal{C}(\overline{s})$, Cauchy-Lipschitz Theorem implies that an extremal trajectory cannot reach the target in this case. Similarly, if we assume that $\lambda_2 < 0$ (which happens only in cases (ii) and (v) of the previous proposition), this implies that u_2 is constant equal to zero, consequently an extremal trajectory starting at s^0 cannot reach the target. It follows that we have:

$$s^0 \in \mathcal{D} \backslash \mathcal{C}(\overline{s}) \implies \lambda_2 > 0,$$

If we assume that ϕ_2 is zero on some time interval $[t_1, t_2]$, $t_1 < t_2$, (case (vii)), then we say that the trajectory has a *singular arc*. It follows that for initial conditions $s^0 \in \mathcal{D} \backslash \mathcal{C}(\overline{s})$, the control law provided by Proposition 4.1 can be simplified into:

- (a) $\phi_2(t) > 0 \implies u(t) = (1, 1)$ and $\lambda_2(t) > 0$,
- (b) $\phi_2(t) = 0 \implies u_1(t) = u_2(t) \in [0, 1]$ and $\lambda_2(t) > 0$ (singular arc),
- (c) $\phi_2(t) < 0 \implies u_1(t) = u_2(t) = 0$.

It follows that when the initial condition is outside of $\mathcal{C}(\overline{s})$, then $u_1 = u_2$. In this case, problem (1.6) can be reduced to a minimum time control problem in the plane with a single input $u := u_1 = u_2$ (such that $u \in \{0, 1\}$ or u is singular from the PMP). System (1.3) becomes:

$$\dot{s} = f(s) + ug(s). \tag{4.8}$$

Now, it is standard that the set of points where the control is singular coincides exactly with the set Δ_{SA} (see e.g. [19]). In this setting, the singular arc can be expressed by the following expressions:

$$\begin{aligned} \det(g(s), [f, g](s)) &= [s_{in} - s_1][\gamma'(s_2)(s_1 - s_2) + \gamma(s_2) - \gamma(s_1) - \mu'(s_1)(s_{in} - s_1)(s_1 - s_2)], \\ &= [s_{in} - s_1][(\gamma'(s_2) - \gamma'(s_1))(s_1 - s_2) + [\mu(s_2) - \mu(s_1)](s_{in} - s_2)], \\ &= [s_{in} - s_1][[\mu(s_2) - \mu(s_1) - \mu'(s_1)(s_1 - s_2)](s_{in} - s_1) + \mu'(s_2)(s_{in} - s_2)(s_1 - s_2)], \end{aligned} \tag{4.9}$$

and thus:

$$\Delta_{SA}(s_1, s_2) := \{(s_1, s_2) \in \mathcal{D} \mid \ \gamma'(s_2)(s_1 - s_2) + \gamma(s_2) - \gamma(s_1) - \mu'(s_1)(s_{in} - s_1)(s_1 - s_2) = 0\}. \tag{4.10}$$

Also, the derivative of the switching function ϕ_2 satisfies (recall that $u_1 = u_2$):

$$\dot{\phi}_2 = \mu'(s_1)(s_{in} - s_1)\phi_2 + \lambda_2 \frac{\det(g(s), [f, g](s))}{s_{in} - s_1}. \tag{4.11}$$

One can check that if $\rho(s_1, s_2) := \mu(s_2)(s_{in} - s_2) - \mu(s_1)(s_1 - s_2)$, then $\det(f(s), g(s)) = (s_{in} - s_1)\rho(s_1, s_2)$, so we get:

$$\Delta_0 = \{(s_1, s_2) \in \mathcal{D} \mid \rho(s_1, s_2) = 0\}, \tag{4.12}$$

which allows to obtain the following expression of the adjoint vector at a switching point.

Lemma 4.1. *If t_0 is a switching point such that $s(t_0) \notin \Delta_0$, we have*

$$\lambda_1(t_0) = \frac{s_1 - s_2}{s_{in} - s_1} \frac{1}{\rho(s_1, s_2)}, \;\; \lambda_2(t_0) = -\frac{1}{\rho(s_1, s_2)}. \tag{4.13}$$

Proof. The proof follows easily from solving the system given by $\phi(t_0) = 0$ and $H = 0$ (recall $\rho(s_1, s_2) \neq 0$). □

Let us recall the clocked form argument based on Green's Theorem in the plane (see e.g. [2, 19, 21, 13]), which allows to compare locally the cost of two different trajectories of (1.3) with $u_1 = u_2$, connecting the same initial point to the same target point.

Theorem 4.1. *Consider two points $s^1, s^2 \in \mathcal{D}$, and two trajectories T_1 (of time t_1) and T_2 (of time t_2) joining s^1 to s^2, such that the trajectory T_1 from s^1 to s^2 followed by the trajectory T_2 from s^2 to s^1 is a positively oriented curve Γ. Moreover, assume that T_1 and T_2 coincide only at the points s^1 and s^2. Let Ω be the open region enclosed by Γ. If $\Omega \cap \Delta_0 = \emptyset$, then:*

$$t_1 - t_2 = \iint_{\Omega} \frac{\det(g(s), [f, g](s))}{\det(f(s), g(s))^2} ds_1 ds_2. \tag{4.14}$$

We insist on the fact that this Theorem answers only locally on the minimal time problem. First notice that whenever admissible trajectories intersect the set of points where Δ_0 is zero, then (4.14) is undefined, and this theorem cannot be applied in order to compare the cost of trajectories. Also, the sign of Δ_{SA} is non-necessarily constant in the domain Ω so that (4.14) may not be helpful in order to prove the optimality of an extremal control. Whenever both sets Δ_0 and Δ_{SA} intersect each other (see the next subsection), this affects the optimal synthesis.

The next lemma allows to exclude extremal trajectories.

Lemma 4.2. *(i) Consider an extremal trajectory such that $u = 1$ on $[0, t_0]$ and $u = 0$ on $[t_0, t_1]$. If $s(t_0) \in \Delta_{SA}^+$, then this trajectory is not optimal.*
(ii) Consider an extremal trajectory such that $u = 0$ on $[0, t_0]$ and $u = 1$ on $[t_0, t_1]$. If $s(t_0) \in \Delta_{SA}^-$, then this trajectory is not optimal.

Proof. The result follows from (4.11). In the first case, we necessarily have $\dot{\phi}_2(t_0) \leq 0$ in contradiction with (4.11) as $s(t_0) \in \Delta_{SA}^+$. The proof of (ii) is similar. □

4.2 Study of singular arcs

In this part, we study singular arcs in the particular case where μ is linear and of type Monod. We call *steady-state singular point* (see [3]) any point $s \in \mathcal{D} \backslash \{(s_{in}, s_{in})\}$ such that:

$$s \in \Delta_0 \cap \Delta_{SA}, \tag{4.15}$$

if this intersection is non-empty. These points are of particular interest in our study. In view of the condition $H = 0$ along any extremal trajectory, the time to reach such a point along a singular arc is infinite provided that the singular control is always admissible. The existence of such points can have consequences on the optimal synthesis.

Proposition 4.2. *If μ is strictly concave, then the singular arc is contained in the sub-domain of $\mathcal{D}$ for which $s_2 > 2s_1 - s_{in}$*

Proof. By strict concavity of μ, we have $\mu(s_1) < \mu(s_2) + \mu'(s_2)(s_1 - s_2)$. Using (4.9), we obtain after some simplifications:

$$\mu(s_1) - \mu(s_2) < \mu'(s_1)(s_{in} - s_1).$$

By strict concavity, we also have $\mu'(s_1)(s_1 - s_2) < \mu(s_1) - \mu(s_2)$. Combining with the previous display yields that:

$$[\mu(s_1) - \mu(s_2)](s_1 - s_2) < \mu'(s_1)(s_{in} - s_1)(s_1 - s_2) < [\mu(s_1) - \mu(s_2)](s_{in} - s_1),$$

which after simplification gives the result (recall that μ is increasing). □

For convenience, we write $\det(g(s),[f,g](s)) = (s_{in}-s_1)\psi(s_1,s_2)$ where:

$$\psi(s_1,s_2) := [\mu(s_2)-\mu(s_1)-\mu'(s_1)(s_1-s_2)](s_{in}-s_1)+\mu'(s_2)(s_{in}-s_2)(s_1-s_2), \tag{4.16}$$

and the zeros of ψ coincide exactly with the singular arc. The expression of a singular control can be obtained by derivating the switching function two times with respect to t:

$$\ddot{\phi} = \langle \lambda, [f,[f,g]]\rangle + u\,\langle \lambda, [g,[f,g]]\rangle\,, \tag{4.17}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^2$, $g(s)^\perp := (-g_2(s), g_1(s))$. If a singular arc is optimal, then we must have $\langle \lambda, [g,[f,g]]\rangle \leq 0$ by Legendre-Clebsch condition (see e.g. [27]). Moreover, if $\langle \lambda, [g,[f,g]]\rangle \neq 0$, then a singular control u_s can be expressed by:

$$u_s = -\frac{\langle \lambda, [f,[f,g]]\rangle}{\langle \lambda, [g,[f,g]]\rangle}, \tag{4.18}$$

where $\lambda = -\frac{-g(s)^\perp}{\mathrm{get}(f(s),g(s))}$. Now, we say that the singular arc is *controllable* provided that $u_s \in [0,1]$. We now give an expression of $\langle \lambda, [f,[f,g]]\rangle$ and $\langle \lambda, [g,[f,g]]\rangle$ taking into account (4.9) and (4.13). First, we have:

$$[f,h] = \begin{pmatrix} -\gamma(s_1)\gamma''(s_1)(s_{in}-s_1)+\gamma'(s_1)[\gamma(s_1)+\gamma'(s_1)(s_{in}-s_1)] \\ -\gamma(s_2)\gamma''(s_2)(s_1-s_2)+\gamma(s_1)\gamma'(s_1)+\gamma'(s_2)[-2\gamma(s_1)+\gamma(s_2)+\gamma'(s_2)(s_1-s_2)], \end{pmatrix}, \tag{4.19}$$

$$[g,h] = \begin{pmatrix} \gamma(s_1)+\gamma'(s_1)+\gamma''(s_1)(s_{in}-s_1)^2 \\ -2\gamma(s_1)+\gamma(s_2)-2\gamma'(s_1)(s_{in}-s_1)+\gamma'(s_2)(s_{in}-s_2)+\gamma''(s_2)(s_1-s_2)^2 \end{pmatrix}. \tag{4.20}$$

It follows that:

$$\begin{aligned}\rho(s_1,s_2)(s_{in}-s_1)\,\langle \lambda, [f,[f,g]]\rangle &= [\gamma(s_2)\gamma''(s_2)-\gamma(s_1)\gamma''(s_1)](s_1-s_2)(s_{in}-s_1) \\ &\quad + [\gamma'(s_1)-\gamma'(s_2)][-2\gamma(s_1)+\gamma(s_2)+\gamma'(s_2)(s_1-s_2](s_{in}-s_1),\end{aligned} \tag{4.21}$$

$$\begin{aligned}\rho(s_1,s_2)(s_{in}-s_1)\,\langle \lambda, [g,[f,g]]\rangle &= [(s_{in}-s_1)\gamma''(s_1)-(s_1-s_2)\gamma''(s_2)](s_1-s_2)(s_{in}-s_1) \\ &\quad + (s_{in}-s_1)[\gamma(s_1)+2\gamma'(s_1)(s_{in}-s_1)-\gamma'(s_2)(s_{in}-s_1)].\end{aligned} \tag{4.22}$$

These expressions allow to compute u_s and can be verified using a symbolic software. Finally, we provide the sign of $\dot{s}$ along a singular arc.

Proposition 4.3. *Assume that the singular arc* (4.9) *defines an increasing function with respect to* s_1 *and that there exists exactly one steady-state singular point* $s^* \neq (s_{in}, s_{in})$ *such that whenever* $s \in \Delta_{SA}$*, we have* $s \in \Delta_0^-$ *(resp.* $s \in \Delta_0^+$*) iff* $s_1 < s_1^*$ *(resp.* $s_1 > s_1^*$*). Then, if at some point* $s \in \Delta_{SA}$ *the singular arc is controllable, then we have* $\dot{s}_{1|_{u=u_s}} > 0$ *if* $s \in \Delta_0^-$ *and* $\dot{s}_{1|_{u=u_s}} < 0$ *if* $s \in \Delta_0^+$*.*

Proof. Take a point $s_0 \in \Delta_0^-$ and assume that the singular arc is controllable at s_0. This means that $u_s \in [0,1$, so the vector $\dot{s}_{1|_{u=u_s}}$ belongs to the positive cone generated by $f(s_0)$ and $f(s_0)+g(s_0)$. Combining the fact that $\mathrm{get}(f(s_0), f(s_0)+g(s_0)) > 0$ and that the singular arc is the graph of an increasing function, we necessarily have $\dot{s}_{1|_{u=u_s}} > 0$ along Δ_{SA}. The proof is the same for a point $s_0 \in \Delta_0^+$. □

Remark 4.2. *The assumptions above can be verified in the case where* μ *is either linear or of type Monod (see the examples below and Fig. 4).*

Combining Proposition 4.3 and Theorem 4.1 yields to the following property. Let s_0, s_0' two points in $\Delta_0^- \cap \Delta_{SA}$ (resp. in $\Delta_0^+ \cap \Delta_{SA}$). Assume that the trajectory with $u=1$ and $u=0$ connecting s_0 to s_0' stays in the set Δ_0^- (resp. Δ_0^+). Let t_1 the time of the singular arc steering s_0 to s_0', and t_2 the time of the trajectory with $u=1$ and $u=0$. Then, we have:

$$s_0, s_0' \in \Delta_0^- \cap \Delta_{SA} \Longrightarrow t_1 - t_2 \leq 0,\;\; s_0, s_0' \in \Delta_0^+ \cap \Delta_{SA} \Longrightarrow t_1 - t_2 \geq 0.$$

Roughly speaking, an optimal trajectory will take advantage of the singular arc below Δ_0 whereas above Δ_0, it is not optimal for a trajectory to stay on a singular arc.

4.2.1 Singular arc when μ is linear

We now consider a particular case where the growth function is given by $\mu(s) = \alpha s$, hence $\gamma(s) = \alpha s(s_{in} - s)$.

Proposition 4.4. *(i) The singular arc is characterized as follows:*

$$\psi(s_1, s_2) = (s_1 - s_2)(2s_1 - s_2 - s_{in}) \text{ and } \Delta_{SA} = \{(s_1, s_2) \in \mathcal{D} \mid s_2 = 2s_1 - s_{in}\}.$$

(ii) There exists exactly one steady-state singular point $s^* := (\frac{2s_{in}}{3}, \frac{s_{in}}{3})$.
(iii) The singular control is given by $u_s := \alpha(s_{in} - s_1)$*, and* $u_s \in [0, 1]$.
(iv) Along the singular arc, we have:

$$\begin{cases} \dot{s}_1 > 0 \text{ iff } s_1 \in [\frac{s_{in}}{2}, \frac{2}{3}s_{in}) \text{ and } \dot{s}_1 < 0 \text{ iff } s_1 \in (\frac{2}{3}s_{in}, s_{in}] \\ \dot{s}_2 > 0 \text{ iff } s_1 \in [0, \frac{s_{in}}{3}) \text{ and } \dot{s}_2 < 0 \text{ iff } s_2 \in (\frac{2s_{in}}{3}, s_{in}] \end{cases} \tag{4.23}$$

(v) The singular arc is controllable provided that $s \neq s^*$*.*
(vi) The adjoint vector is given by

$$\lambda_1 = -\lambda_2 = -\frac{1}{\alpha(s_{in} - s_1)(-3s_1 + 2s_{in})}, \ s_1 \neq \frac{2s_{in}}{3}.$$

Proof. The proof of (i) and (ii) is straightforward. Notice that we have $s_{in} - s_1 = s_1 - s_2$ along the singular arc. The expression of u_s follows from (1.3) using $\dot{s}_2 = 2\dot{s}_1$ which proves (iii). Now replacing u_s into (1.3) gives the closed-loop system:

$$\begin{cases} \dot{s}_1 = \alpha(s_{in} - s_1)(2s_{in} - 3s_1), \\ \dot{s}_2 = \alpha(s_{in} - s_1)(s_{in} - 3s_2). \end{cases} \tag{4.24}$$

This proves (iv) and (v) follows from the fact that at s^* we have both vector fields f and g are parallel to $(s_{in} - s_1, s_1 - s_2)$ which is not collinear to the singular arc. The proof of (vi) follows from (4.13) $\square$

Notice that a singular trajectory cannot cross s^* which is an equilibrium of (4.24). In other words, a singular extremal trajectory cannot reach s^* in finite time. The time of a singular extremal trajectory from a substrate concentration s_1^0 will converge to $+\infty$ whenever s goes to s^*. This case is illustrated on Fig. 4.

4.2.2 Singular arc when μ is Monod

We now consider a particular case where the growth function is given by $\mu(s) = \frac{\bar{\mu}s}{k+s}$. The situation is quite similar to the linear case, but the expression of Δ_0, Δ_{SA} and u_s are more delicate, and we have used a symbolic software in order to verify the next proposition.

Proposition 4.5. *(i) There exists* $\check{s}_1 \in (0, s_{in})$ *and a* C^1*-mapping* $F_{AS} : [\check{s}_1, s_{in}) \to [0, s_{in})$ *which is increasing and such that* $(s_1, s_2) \in \mathcal{D}$ *satisfies* $\psi(s_1, s_2) \geq 0$ *if and only if* $s_2 \geq F_{AS}(s_1)$*, where :*

$$F_{AS}(s_1) := \frac{1}{2(s_{in} - s)} \left[-s_{in}s_1 - k^2 + ks_1 - 3ks_{in} + \sqrt{(k + s_1)^2(s_{in} + k)(5s_{in} + k - 4s_1)} \right], \tag{4.25}$$

and we have:

$$\check{s}_1 := \frac{-2k - s_{in} + k\sqrt{4k^2 + 8ks_{in} + 5s_{in}^2}}{2(s_{in} + k)}.$$

(ii) There exists a continuous function $F_0 : [0, s_{in}] \to [0, s_{in}]$ *such that* $(s_1, s_2) \in \Delta_0^+$ *iff* $s_2 \geq F_0(s_1)$*, and we have:*

$$F_0(s_1) = -\frac{1}{2k} \left[s_1^2 - ks_1 - ks_{in} - s_{in}s_1 + \sqrt{(s_{in} - s_1)(ks_{in} + s_{in}s_1 - s_1^2 + 3ks_1)} \right], \ s_1 \in (0, s_{in}). \tag{4.26}$$

For particular values of the parameters, we can check that there exists exactly one singular point s^* such that $g(s_1^*) = h(s_1^*)$. Moreover, if u_s is admissible, we have that $\dot{s}_1 > 0$ (resp. $\dot{s}_2 < 0$) iff $s_1 < s_1^*$ (resp. $s_1 > s_1^*$), and $\dot{s}_2 > 0$ (resp. $\dot{s}_2 < 0$) iff $s_2 < s_2^*$ (resp. $s_2 > s_2^*$), see Proposition 4.3 and Fig. 4.

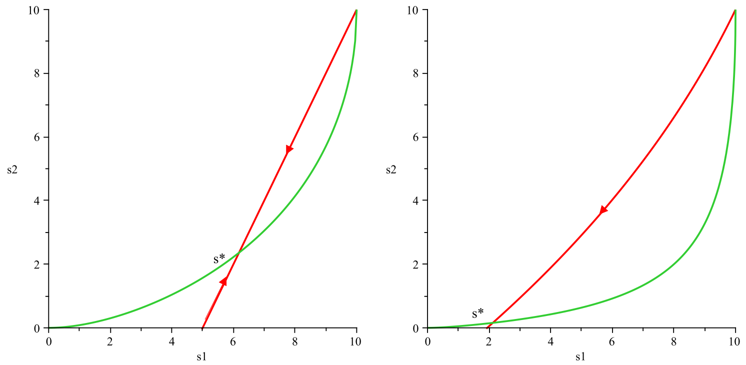

Figure 4: *Picture left*: plot of the singular arc Δ_{SA} (in red) and of Δ_0 (in green) in the linear case: $\mu(s) = s$, $s_{in} = 10$. *Picture right*: plot of the singular arc Δ_{SA} (in red) and of Δ_0 (in green) in the Monod case: $\mu(s) = \frac{s}{5+s}$, $s_{in} = 10$.

4.3 Optimal synthesis for a target in $\Delta_0^- \cup (\Delta_0^+ \cap \mathcal{R}^-)$

Recall that we have the following partition of $\mathcal{D}$:

$$\mathcal{D} = \Delta_{SA}^- \cup \Delta_{SA} \cup \Delta_{SA}^+.$$

Also, one has $s \in \Delta_{SA}^-$ (resp. $s \in \Delta_{SA}^+$) if and only if $s_2 > F_{AS}(s_1)$ (resp. $s_2 < F_{AS}(s_1)$). Next we assume that the singular arc meets the set of non-local-controllability at exactly one point:

$$\{s^*\} = \Delta_0 \cap \Delta_{SA}. \tag{4.27}$$

Definition 4.1. *For $s \in \mathcal{D} \backslash \mathcal{C}(\overline{s})$, we define the singular arc strategy (SAS) as follows:*

$$u = \begin{cases} 1 & \Longleftrightarrow \; s \in \Delta_{SA}^-, \\ u_s & \Longleftrightarrow \; s \in \Delta_{SA}, \\ 0 & \Longleftrightarrow \; s \in \Delta_{SA}^+. \end{cases} \tag{4.28}$$

In other words, the singular arc strategy consists in a most rapid approach (see e.g. [28]) toward the singular arc (it is not necessarily reached if $s_0 \in \Delta_{SA}^-$).

Lemma 4.3. *Let us take $s_0 \in \mathcal{D}$, consider the mapping $t \longmapsto \sigma(t) := s(t, s_0, (1, 1))$, and assume that the singular arc is controllable. Then, if there exists $t_0 \geq 0$ such that $\sigma(t_0) \in \Delta_{SA} \cap \Delta_0^-$, and $\sigma(t) \in \Delta_{SA}^-$ for $t \in [t_0, t_0 + \varepsilon)$ for some $\varepsilon > 0$, then for $t > t_0$ we have $\sigma(t) \in \Delta_{SA}^-$ provided that $\sigma(t) \in \mathcal{D}$.*

Proof. If the assertion is false, there exists $t_1 > t_0$ such that $\sigma(t_1) \in \Delta_{SA} \cap \mathcal{D}$. Without any loss of generality, we may assume that $t_1 := \inf\{t > t_0 \;\mid \sigma(t) \in \Delta_{SA} \cap \mathcal{D}\}$, which implies that $\sigma(t) \in \Delta_{SA}^-$ for any $t \in (t_0, t_1)$. At the point $\sigma(t_0)$, let us consider the extended set of velocities:

$$V(\sigma(t_0)) := \{f(\sigma(t_0)) + ug(\sigma(t_0)) \mid \quad 0 \leq u \leq 1\}.$$

As the singular arc is controllable, the vector $f(\sigma(t_0)) + u_s(\sigma(t_0))g(\sigma(t_0))$ is in $V(\sigma(t_0))$. Now, $f(\sigma(t_0))$ is vector pointing in the set Δ_{SA}^- and from the fact that $\sigma(t) \in \Delta_{SA}^-$ for any $t \in (t_0, t_1)$, the vector $g(\sigma(t_0))$ is also pointing in the set Δ_{SA}^-. Therefore, as $V(\sigma(t_0))$ is convex, we obtain a contradiction with $f(\sigma(t_0)) + u_s(\sigma(t_0))g(\sigma(t_0)) \in V(\sigma(t_0))$. $\square$

We have the following result.

Proposition 4.6. *Assume that the growth function satisfies (H1)-(H2)-(H3). Let us be given a target point $\overline{s}$ in the set Δ_0^- or in $\Delta_0^+ \cap \mathcal{R}^-$, and assume that $s^* \in \mathcal{C}(\overline{s})$. Then, provided that the singular arc is controllable and that the boundary of $\mathcal{C}(\overline{s})$ intersects the singular arc at at most one point $s_c \neq (s_{in}, s_{in})$, the optimal strategy to steer any initial condition in $\mathcal{D}\backslash\mathcal{C}(\overline{s})$ to the target point is the singular arc strategy until reaching $\partial\mathcal{C}(\overline{s})$.*

Proof. Lemma 4.2 allows to exclude extremal trajectories that have one switching before reaching the singular arc (if it is reachable). This implies that an extremal trajectory cannot switch in the set Δ_{SA}^{+} from $u = 1$ to $u = 0$ or in the set Δ_{SA}^{-} from $u = 0$ to $u = 1$. Now, whenever an optimal trajectory reaches the singular arc, the strategy remains singular until reaching the point s_c by applying the clocked form property. $\square$

From the study of optimal trajectories in the set $\mathcal{C}(\overline{s})$, we have that after reaching the set $\mathcal{C}(\overline{s})$, an optimal trajectory necessarily satisfies $u = 0$ until $\overline{s}$ or $u = 1$ until $\overline{s}$.

Remark 4.3. *(i) The controllability assumption in Proposition 4.6 is always satisfied in the linear case (see Proposition 4.4 (iii)). We can check numerically that it is satisfied in the Monod case using* (4.18).
(ii) If the boundary of $\mathcal{C}(\overline{s})$ intersects the singular arc for the control $u = 0$ (see Proposition above), then by convexity (see Proposition 2.7 (iii)), there exists at most one point $s_c \neq (s_{in}, s_{in})$ such that $s_c \in \partial\mathcal{C}(\overline{c}) \cap \Delta_{SA}$.
(iii) If the target is in Δ_0^+, but $s^* \in \mathcal{C}(\overline{s})$, the target is possibly non-locally controllable, nevertheless condition $s^* \in \mathcal{C}(\overline{s})$ implies that the singular strategy is admissible so that Proposition 4.6 still holds in this case.

Proposition 4.6 is illustrated when μ is linear in Fig. 5,6,7. Under the condition $s^* \in \mathcal{C}(\bar{s})$, there are three kinds of optimal synthesis $u_1^\star = u_2^\star = u^\star$ in the domain $\mathcal{D} \setminus \mathcal{C}(\bar{s})$, that differ from the possible sequences of switching, given in Table 1. When $\bar{s} \in \Delta_{SA}^+$, we consider s_c the intersection point of Δ_{AS} with $\partial\mathcal{C}(\bar{s})$, that we assume to be unique, and define the parametrized curve $c(\cdot)$ as

$$c : \left| \begin{array}{ccc} [0, \tilde{t}_1'(s_c)) & \mapsto & \mathcal{D} \\ t & \to & s(-t, s_c, (1,1)) \end{array} \right.$$

We then define the sets $\Delta_c^+ := c_{\geq}$ and $\Delta_c^- := \mathcal{D} \setminus \Delta_c^+$. It follows that if the initial point is such that $s_0 \in \Delta_{SA}^+ \cap \Delta_c^+$ (see Fig. 5), then an optimal trajectory reaches the boundary of $\mathcal{C}$ before reaching the singular arc. The structure of the optimal control for different initial conditions is summarized in Table 1, and the different values of $\bar{s}$ are given in Table 2.

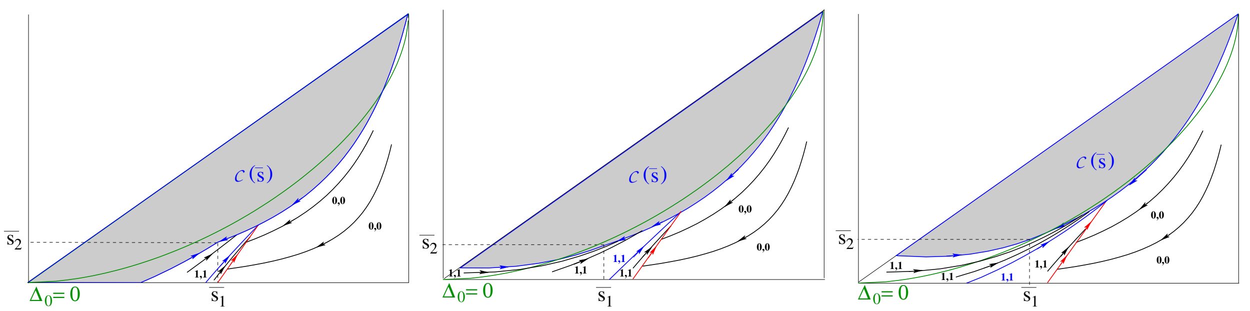

Figure 5: Optimal trajectories for a target $\bar{s} \in \Delta_{SA}^+$. *Picture left*: $\bar{s} \in \mathcal{R}^- \cap \Delta_0^-$. *Picture in the middle*: $\bar{s} \in \mathcal{R}^+ \cap \Delta_0^+$. *Picture right*: $\bar{s} \in \mathcal{R}^+ \cap \Delta_0^-$.

5 Conclusion

In this work, we have analyzed a minimal time control problem by decomposing the state space into a set $\mathcal{C}(\overline{s})$ depending on the target $\overline{s}$ and its complementary in $\mathcal{D}$. Whereas in the domain $\mathcal{C}(\overline{s})$, we could completely solve the optimal control problem by a direct computation of the value function, the analysis of optimal trajectories in the complementary of $\mathcal{C}(\overline{s})$ is more delicate to handle in view of the non-controllability curve Δ_0. We have provided a complete optimal synthesis of the problem in the case where the target is below Δ_0, the singular arc is controllable and whenever the steady-state singular point is in $\mathcal{C}(\overline{s})$. The case where the singular steady-state point could be outside $\mathcal{D}\backslash\mathcal{C}(\overline{s})$ is more difficult as the singular arc strategy is no longer optimal, and it will deserve further investigations.

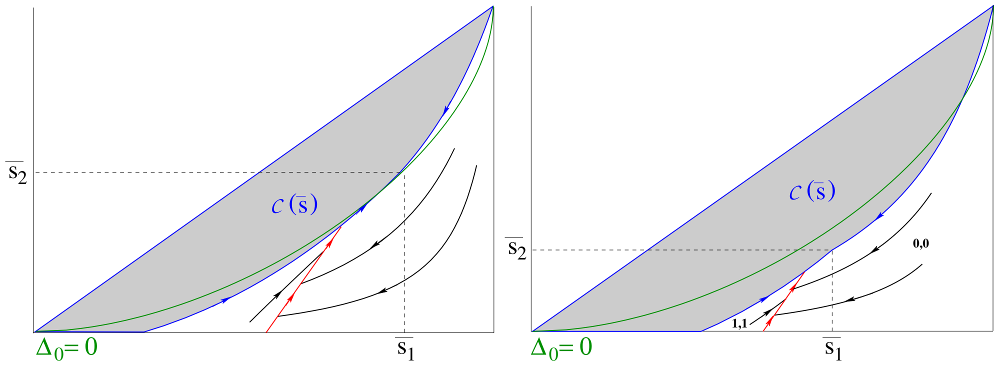

Figure 6: Optimal trajectories for a target $\bar{s} \in \Delta_{SA}^{-}$ such that $\Delta_{SA} \not\subset \mathcal{C}(\bar{s})$. *Picture left*: $\bar{s} \in \mathcal{R}^{-} \cap \Delta_0^{+}$. *Picture right*: $\bar{s} \in \mathcal{R}^{-} \cap \Delta_0^{-}$

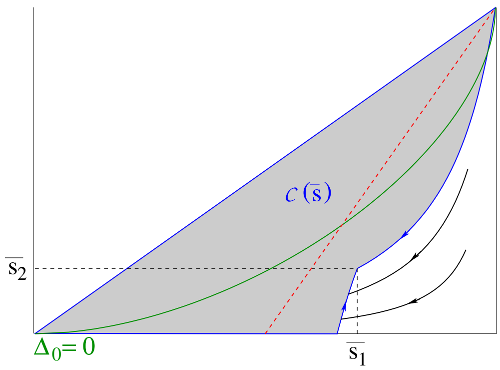

Figure 7: Optimal trajectories in the case $\Delta_{SA} \subset \mathcal{C}(\bar{s})$.

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Target	Optimal sequence of switching	Figure
$\bar{s} \in \Delta_{SA}^{+}$	$u^\star = \left\lvert \begin{array}{ll} 0, u_s, 0 & \text{if } s_0 \in \Delta_{SA}^{-} \\ 1, u_s, 0 & \text{if } s_0 \in \Delta_{SA}^{+} \cap \Delta_c^{-} \\ 1, 0 & \text{if } s_0 \in \Delta_{SA}^{+} \cap \Delta_c^{+} \end{array} \right.$	Fig. 5
$\bar{s} \in \Delta_{SA}^{-}$ and $\Delta_{SA} \not\subset \mathcal{C}(\bar{s})$	$u^\star = \left\lvert \begin{array}{ll} 0, u_s, 1 & \text{if } s_0 \in \Delta_{SA}^{-} \\ 1, u_s, 1 & \text{if } s_0 \in \Delta_{SA}^{+} \end{array} \right.$	Fig. 7
$\Delta_{SA} \subset \mathcal{C}(\bar{s})$	$u^\star = 0, 1$	Fig. 6

Table 1: Structure of the optimal control for different choice of the target $\bar{s}$.

Figure	$\bar{s}$	α	S_{in}	$u_{\max}$
5 left	$(1, 0.3)$	0.5	2	1
5 middle	$(0.85, 0.26)$	0.5	2	1
5 right	$(0.9, 0.32)$	0.5	2	1
7 left	$(1.6, 0.985)$	0.5	2	1
7 right	$(1.3, 0.5)$	0.1	2	1
6	$(1.4, 0.4)$	0.5	2	1

Table 2: Numerical values for figures 5,7,6.

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