CONSTRUCTIBLE CHARACTERS AND *b*-INVARIANT

by

CÉDRIC BONNAFÉ

Let (W,S) be a finite Coxeter system and let $\varphi : S \to \mathbb{R}_{>0}$ be a *weight function* that is, a map such that $\varphi(s) = \varphi(t)$ whenever s and t are conjugate in W. Associated with this datum, G. Lusztig has defined [**Lu3**, §22] a notion of *constructible characters* of W: it is conjectured that a character is constructible if and only if it is the character afforded by a Kazhdan-Lusztig left cell (defined using the weight function φ). These constructible characters depend heavily on φ so we will call them the *φ-constructible characters of W*: the set of φ-constructible characters will be denoted by $\mathrm{Cons}_\varphi^{\mathrm{Lus}}(W)$. We shall also define a graph $\mathscr{G}_{W,\varphi}^{\mathrm{Lus}}$ as follows: the vertices of $\mathscr{G}_{W,\varphi}^{\mathrm{Lus}}$ are the irreducible characters and two irreducible characters χ and χ' are joined in this graph if there exists a φ-constructible character γ of W such that χ and χ' both occur as constituents of γ. The connected components of $\mathscr{G}_{W,\varphi}^{\mathrm{Lus}}$ (viewed as subsets of $\mathrm{Irr}(W)$) will be called the *Lusztig φ-families*: the set of Lusztig φ-families will be denoted by $\mathrm{Fam}_\varphi^{\mathrm{Lus}}(W)$. If $\mathscr{F} \in \mathrm{Fam}_\varphi^{\mathrm{Lus}}(W)$, we denote by $\mathrm{Cons}_\varphi^{\mathrm{Lus}}(\mathscr{F})$ the set of φ-constructible characters of W all of whose irreducible components belong to $\mathscr{F}$.

On the other hand, using the theory of rational Cherednik algebras at $t = 0$ and the geometry of the Calogero-Moser space associated with (W,φ), R. Rouquier and the author (see [**BoRo1**] and [**BoRo2**]) have defined a notion of *Calogero-Moser φ-cells* of W, a notion of *Calogero-Moser φ-cellular characters* of W (whose set is denoted by $\mathrm{Cell}_\varphi^{\mathrm{CM}}(W)$) and a notion of *Calogero-Moser φ-families* (whose set is denoted by $\mathrm{Fam}_\varphi^{\mathrm{CM}}(W)$).

Conjecture (see [BoRo1], [BoRo2] and [GoMa]). *With the above notation,*

$$\mathrm{Cons}_\varphi^{\mathrm{Lus}}(W) = \mathrm{Cell}_\varphi^{\mathrm{CM}}(W) \quad \textit{and} \quad \mathrm{Fam}_\varphi^{\mathrm{Lus}}(W) = \mathrm{Fam}_\varphi^{\mathrm{CM}}(W)$$

for every weight function $\varphi : S \to \mathbb{R}_{>0}$.

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The statement about families in this conjecture holds for classical Weyl groups thanks to a case-by-case analysis relying on [**Lu3**, §22] (for the computation of Lusztig φ-families), [**GoMa**] (for the computation of Calogero-Moser φ-families in type A and B) and [**Be2**] (for the computation of the Calogero-Moser φ-families in type D). It also holds whenever $|S| = 2$ (see [**Lu3**, §17 and Lemma 22.2] and [**Be1**, §6.10]).

The statement about constructible characters is much more difficult to establish, as the computation of Calogero-Moser φ-cellular characters is at that time out of reach: it has been proved whenever the Caloger-Moser space associated with (W,S,φ) is smooth [**BoRo2**, Theorem 14.4.1] (it has also been checked if W is of type B_2...).

Our aim in this paper is to show that this conjecture is compatible with properties of the b-invariant (as defined below). With each irreducible character χ of W is associated its *fake degree* $f_\chi(\mathbf{t})$, using the invariant theory of W (see for instance [**BoRo2**, Definition 1.5.7]). Let us denote by b_χ the valuation of $f_\chi(\mathbf{t})$: b_χ is called the *b-invariant* of χ. For instance, $b_1 = 0$ and b_ε is the number of reflections of W (here, $\varepsilon : W \to \{1,-1\}$ denotes the sign character). Also, $b_\chi = 1$ if and only if χ is an irreducible constituent of the canonical reflection representation of W. The following result is proved in [**BoRo2**, Theorems 9.6.1 and 12.3.14]:

Theorem CM. *Let $\varphi : S \to \mathbb{R}_{>0}$ be a weight function. Then:*

(a) *If $\mathscr{F} \in \mathrm{Fam}_\varphi^{\mathrm{CM}}(W)$, then there exists a unique $\chi \in \mathscr{F}$ with minimal b-invariant.*

(b) *If $\gamma \in \mathrm{Cons}_\varphi^{\mathrm{CM}}(W)$, then there exists a unique irreducible constituent χ of γ with minimal b-invariant.*

The next theorem is proved in [**Lu2**, Theorem 5.25 and its proof] (see also [**Lu1**] for the first occurence of the *special* representations):

Theorem (Lusztig). *Assume that φ is **constant**. Then:*

(a) *If $\mathscr{F} \in \mathrm{Fam}_\varphi^{\mathrm{Lus}}(W)$, then there exists a unique $\chi_\mathscr{F} \in \mathscr{F}$ with minimal b-invariant ($\chi_\mathscr{F}$ is called the **special** character of $\mathscr{F}$).*

(b) *If $\gamma \in \mathrm{Cons}_\varphi^{\mathrm{Lus}}(\mathscr{F})$, then $\chi_\mathscr{F}$ is an irreducible constituent of γ (and, of course, among the irreducible constituents of γ, $\chi_\mathscr{F}$ is the unique one with minimal b-invariant).*

It turns out that, for general φ, there might exist Lusztig φ-families $\mathscr{F}$ such that no element of $\mathscr{F}$ occurs as an irreducible constituent of *all* the φ-constructible characters in $\mathrm{Cons}_\varphi^{\mathrm{Lus}}(\mathscr{F})$ (this already occurs in type B_3, and the reader can also check this fact in type F_4, using the tables given by Geck [**Ge**, Table 2]). Nevertheless, we will prove in this paper the following result, which is compatible with the above conjecture and the above theorems:

Theorem L. *Let $\varphi : S \to \mathbb{R}_{>0}$ be a weight function. Then:*

(a) *If $\mathscr{F} \in \mathrm{Fam}_\varphi^{\mathrm{Lus}}(W)$, then there exists a unique $\chi \in \mathscr{F}$ with minimal b-invariant.*

(b) *If $\gamma \in \mathrm{Cons}_\varphi^{\mathrm{Lus}}(W)$, then there exists a unique irreducible constituent χ of γ with minimal b-invariant.*

The proof of Theorem CM is general and conceptual, while our proof of Theorem L goes through a case-by-case analysis, based on Lusztig's description of φ-constructible characters and Lusztig φ-families [**Lu3**, §22].

REMARK 0 - Let γ_χ denote the coefficient of $\mathbf{t}^{b_\chi}$ in $F_\chi(\mathbf{t})$. Then it has been noticed by Lusztig [**Lu1**, §2, Page 325] that $\gamma_\chi = 1$ whenever χ is special.

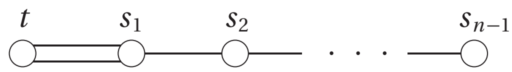
As the only irreducible Coxeter systems affording possibly unequal parameters are of type $I_2(2m)$, F_4 or B_n, and as $\gamma_\chi = 1$ for any character χ in these groups, we can conclude that, in general (equal or unequal parameters), $\gamma_\chi = 1$ for all the characters χ with minimal b-invariant constructed in Theorem L (for both (a) and (b)).

The same property holds for the characters χ with minimal b-invariant constructed in Theorem CM (in this case, the proof is again general and conceptual [**BoRo2**]). ■

1. Proof of Theorem L

1.A. Reduction. — It is easily seen that the proof of Theorem L may be reduced to the case where (W,S) is irreducible. If W is of type A_n, D_n, E_6, E_7, E_8, H_3 or H_4, then φ is necessarily constant and Theorem L follows immediately from Lusztig's Theorem. If W is dihedral, then Theorem L is easily checked using [**Lu3**, §17 and Lemma 22.2]. If W is of type F_4, then Theorem L follows from inspection of [**Ge**, Table 2]. Therefore, this shows that we may, and we will, assume that W is of type B_n, with $n \geqslant 2$. Write $S = \{t, s_1, s_2, \dots, s_{n-1}\}$ in such a way that the Dynkin diagram of (W,S) is

(#)



Write $b = \varphi(t)$ and $a = \varphi(s_1) = \varphi(s_2) = \cdots = \varphi(s_{n-1})$. If $b \notin a\mathbb{N}^*$, then $\mathrm{Cons}_\varphi^{\mathrm{Lus}}(W) = \mathrm{Irr}(W)$ (see [**Lu3**, Proposition 22.25]) and Theorem L becomes obvious. So we may assume that $b = ra$ with $r \in \mathbb{N}^*$. To summarize:

> **Hypothesis.** *From now on, and until the end of this section, we will assume that (W,S) is of type B_n, with $n \geqslant 2$, that $S = \{t, s_1, s_2, \dots, s_{n-1}\}$ is such that the Dynkin diagram of (W,S) is given by (#), that $\varphi(t) = r\varphi(s_1) = r\varphi(s_2) = \cdots = r\varphi(s_{n-1}) = 1$ with $r \in \mathbb{N}^*$.*

1.B. Admissible involutions. — Let $l \geqslant 0$ and let Z be a totally ordered set of size $2l+r$. We shall define by induction on l what is an r-admissible involution of Z. Let $\iota : Z \to Z$ be an involution. Then ι is said *r-admissible* if it has r fixed points and, if $l \geqslant 1$, there exist two consecutive elements b and c of Z such that $\iota(b)=c$ and the restriction of ι to $Z \setminus \{b,c\}$ is r-admissible.

Note that, if ι is an r-admissible involution and if $\iota(b)=c>b$ and $\iota(z)=z$, then $z<b$ or $z>c$ (this is easily proved by induction on $|Z|$).

1.C. Symbols. — We shall denote by $\mathbf{Sym}_k(r)$ the set of *symbols* $\Lambda = \binom{\beta}{\gamma}$ where $\beta = (\beta_1 < \beta_2 < \cdots < \beta_{k+r})$ and $\gamma = (\gamma_1 < \gamma_2 < \cdots < \gamma_k)$ are increasing sequences of *non-zero* natural numbers. We set

$$|\Lambda| = \sum_{i=1}^{k+r} (\beta_i - i) + \sum_{j=1}^{k} (\gamma_j - j)$$

and
$$\mathbf{b}(\Lambda) = \sum_{i=1}^{k+r} (2k+2r-2i)(\beta_i - i) + \sum_{j=1}^{k} (2k+1-2j)(\gamma_j - j).$$

The number $\mathbf{b}(\Lambda)$ will be called the $\mathbf{b}$-*invariant* of Λ. For simplifying our arguments, we shall define

$$\nabla_{k,r} = \sum_{i=1}^{k+r} (2k+2r-2i)i + \sum_{j=1}^{k} (2k+1-2j)j$$

so that

$$\mathbf{b}(\Lambda) = \sum_{i=1}^{k+r} (2k+2r-2i)\beta_i + \sum_{j=1}^{k} (2k+1-2j)\gamma_j - \nabla_{k,r}.$$

By abuse of notation, we denote by $\beta \cap \gamma$ the set $\{\beta_1, \beta_2, \ldots, \beta_{k+r}\} \cap \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ and by $\beta \cup \gamma$ the set $\{\beta_1, \beta_2, \ldots, \beta_{k+r}\} \cup \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$. We also set $\beta + \gamma = (\beta \cup \gamma) \setminus (\beta \cap \gamma)$.

Now, let $\mathbf{z}'(\Lambda) = (\beta_1, \beta_2, \ldots, \beta_r, \gamma_1, \beta_{r+1}, \gamma_2, \beta_{r+2}, \ldots, \gamma_k, \beta_{r+k})$. We shall write

$$\mathbf{z}'(\Lambda) = (z'_1(\Lambda), z'_2(\Lambda), \cdots, z'_{2k+r}(\Lambda)),$$

so that

$$\begin{aligned} \mathbf{b}(\Lambda) &= \sum_{i=1}^{r} (2k+2r-2i) z'_i(\Lambda) + \sum_{i=r+1}^{2k+r} (2k+r-i) z'_i(\Lambda) - \nabla_{k,r} \\ (\clubsuit) \qquad &= \sum_{i=1}^{r} (r-i) z'_i(\Lambda) + \sum_{i=1}^{2k+r} (2k+r-i) z'_i(\Lambda) - \nabla_{k,r} \\ &= \sum_{i=1}^{r-1} \Bigl(\sum_{j=1}^{i} z'_j(\Lambda)\Bigr) + \sum_{i=1}^{2k+r-1} \Bigl(\sum_{j=1}^{i} z'_j(\Lambda)\Bigr) - \nabla_{k,r}. \end{aligned}$$

1.D. Families of symbols. — We denote by $\mathbf{z}(\Lambda)$ the sequence $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_{2k+r}$ obtained after rewriting the sequence $(\beta_1,\beta_2,\ldots,\beta_{k+r},\gamma_1,\gamma_2,\ldots,\gamma_k)$ in non-decreasing order.

REMARK 1 - Note that the sequence $\mathbf{z}'(\Lambda)$ determines the symbol Λ, contrarily to the sequence $\mathbf{z}(\Lambda)$. However, $\mathbf{z}(\Lambda)$ determines completely $|\Lambda|$ thanks to the formula $|\Lambda| = \sum_{z\in\mathbf{z}(\Lambda)} z - r(r+1)/2 - (k+r)(k+r+1)/2$. □

We say that two symbols $\Lambda = \begin{pmatrix}\beta\\ \gamma\end{pmatrix}$ and $\Lambda' = \begin{pmatrix}\beta'\\ \gamma'\end{pmatrix}$ in $\mathbf{Sym}_k(r)$ *are in the same family* if $\mathbf{z}(\Lambda) = \mathbf{z}(\Lambda')$. Note that this is equivalent to say that $\beta\cap\gamma = \beta'\cap\gamma'$ and $\beta\cup\gamma = \beta'\cup\gamma'$. If $\mathscr{F}$ is the family of Λ, we set $X_{\mathscr{F}} = \beta\cap\gamma$ and $Z_{\mathscr{F}} = \beta\dot{+}\gamma$: note that $X_{\mathscr{F}}$ and $Z_{\mathscr{F}}$ depend only on $\mathscr{F}$ (and not on the particular choice of $\Lambda\in\mathscr{F}$).

If ι is an r-admissible involution of $Z_{\mathscr{F}}$, we denote by $\mathscr{F}_\iota$ the set of symbols $\Lambda = \begin{pmatrix}\beta\\ \gamma\end{pmatrix}$ in $\mathscr{F}$ such that $|\beta\cap\omega| = 1$ for all ι-orbits ω.

1.E. Lusztig families, constructible characters. — Let $\Lambda \in \mathbf{Sym}_k(r)$ be such that $|\Lambda| = n$. Let $\mathrm{Bip}(n)$ be the set of bipartitions of n. We set

$$\boldsymbol{\lambda}_1(\Lambda) = (\beta_{k+r} - (k+r) \geqslant \cdots \geqslant \beta_2 - 2 \geqslant \beta_1 - 1),$$

$$\boldsymbol{\lambda}_2(\Lambda) = (\gamma_k - k \geqslant \cdots \geqslant \gamma_2 - 2 \geqslant \gamma_1 - 1)$$

and

$$\boldsymbol{\lambda}(\Lambda) = (\boldsymbol{\lambda}_1(\Lambda), \boldsymbol{\lambda}_2(\Lambda)).$$

Then $\boldsymbol{\lambda}(\Lambda)$ is a bipartition of n. We denote by χ_Λ the irreducible character of W denoted by $\chi_{\boldsymbol{\lambda}(\Lambda)}$ in [**Lu3**, §22] or in [**GePf**, §5.5.3]. Then [**GePf**, §5.5.3]

$$(\diamondsuit) \qquad b_{\chi_\Lambda} = \mathbf{b}(\Lambda).$$

With these notation, Lusztig described the φ-constructible characters in [**Lu3**, Proposition 22.24], from which the description of Lusztig φ-families follow by using [**Lu3**, Lemma 22.22]:

Theorem 2 (Lusztig). *Let $\mathscr{F}_{\mathrm{Lus}}$ be a Lusztig φ-family and let $\gamma \in \mathrm{Cons}_\varphi^{\mathrm{Lus}}(\mathscr{F}_{\mathrm{Lus}})$. If we choose k sufficiently large, then:*

(a) *There exists a family $\mathscr{F}$ of symbols in $\mathbf{Sym}_k(r)$ such that*

$$\mathscr{F}_{\mathrm{Lus}} = \{\chi_\Lambda \mid \Lambda\in\mathscr{F}\}.$$

(b) *There exists an r-admissible involution ι of $Z_{\mathscr{F}}$ such that*

$$\gamma = \sum_{\Lambda\in\mathscr{F}_\iota} \chi_\Lambda.$$

Definition 3. *The symbol* Λ *is said* ***special*** *if* $\mathbf{z}(\Lambda)=\mathbf{z}'(\Lambda)$.

REMARK 4 - According to Remark 1, there is a unique special symbol in each family. It will be denoted by $\Lambda_{\mathscr{F}}$. Finally, note that, if Λ, Λ' belong to the same family, then $|\Lambda|=|\Lambda'|$. □

Now, Theorem L follows from Theorem 2, Formula (♢) and the following next Theorem:

Theorem 5. *Let* $\mathscr{F}$ *be a family of symbols in* $\mathbf{Sym}_k(r)$, *let* ι *be an* r*-admissible involution of* $Z_{\mathscr{F}}$ *and let* $\Lambda\in\mathscr{F}$. *Then:*

(a) $\mathbf{b}(\Lambda)\geqslant\mathbf{b}(\Lambda_{\mathscr{F}})$ *with equality if and only if* $\Lambda=\Lambda_{\mathscr{F}}$.

(b) *There is a unique symbol* $\Lambda_{\mathscr{F},\iota}$ *in* $\mathscr{F}_\iota$ *such that, if* $\Lambda\in\mathscr{F}_\iota$, *then* $\mathbf{b}(\Lambda)\geqslant\mathbf{b}(\Lambda_{\mathscr{F},\iota})$, *with equality if and only if* $\Lambda=\Lambda_{\mathscr{F},\iota}$.

1.F. Proof of Theorem 5(a). — First, note that $\mathbf{z}(\Lambda)=\mathbf{z}(\Lambda_{\mathscr{F}})=\mathbf{z}'(\Lambda_{\mathscr{F}})$. As $\mathbf{z}'(\Lambda)$ is a permutation of the non-decreasing sequence $\mathbf{z}'(\Lambda_{\mathscr{F}})$, we have

$$\sum_{j=1}^{i} z'_j(\Lambda)\geqslant\sum_{j=1}^{i} z'_j(\Lambda_{\mathscr{F}})$$

for all $i\in\{1,2,\cdots,2k+r\}$. So, it follows from (♣) that

$$\mathbf{b}(\Lambda)-\mathbf{b}(\Lambda_{\mathscr{F}})=\sum_{i=1}^{r-1}\Bigl(\sum_{j=1}^{i}\bigl(z'_j(\Lambda)-z'_j(\Lambda_{\mathscr{F}})\bigr)\Bigr)+\sum_{i=1}^{2k+r-1}\Bigl(\sum_{j=1}^{i}\bigl(z'_j(\Lambda)-z'_j(\Lambda_{\mathscr{F}})\bigr)\Bigr).$$

So $\mathbf{b}(\Lambda)\geqslant\mathbf{b}(\Lambda_{\mathscr{F}})$ with equality only whenever $\sum_{j=1}^{i} z'_j(\Lambda)=\sum_{j=1}^{i} z'_j(\Lambda_{\mathscr{F}})$ for all $i\in\{1,2,\dots,2k+r\}$. The proof of Theorem 5(a) is complete.

1.G. Reduction for the proof of Theorem 5(b). — First, assume that $X_{\mathscr{F}}\neq\varnothing$. Let $b\in X_{\mathscr{F}}$ and let $\Lambda=\begin{pmatrix}\beta\\ \gamma\end{pmatrix}\in\mathscr{F}$. Then $b\in\beta\cap\gamma=X_{\mathscr{F}}$ and we denote by $\beta[b]$ the sequence obtained by removing b to β. Similarly, let $\Lambda[b]=\begin{pmatrix}\beta[b]\\ \gamma[b]\end{pmatrix}$.

Then $\Lambda[b]\in\mathbf{Sym}_{k-1}(r)$ and

$$(\heartsuit)\qquad \mathbf{b}(\Lambda)=\mathbf{b}(\Lambda[b])+\nabla_{k,r}-\nabla_{k-1,r}+b\Bigl(4k+2r+1-\sum_{\substack{z\in\mathbf{z}(\Lambda)\\ z\leqslant b}}2\Bigr)+2\sum_{\substack{z\in\mathbf{z}(\Lambda)\\ z<b}}z.$$

Proof of (♡). Let i_0 and j_0 be such that $\beta_{i_0}=b$ and $\gamma_{j_0}=b$. Then

$$\mathbf{b}(\Lambda)-\mathbf{b}(\Lambda[b])=\nabla_{k,r}-\nabla_{k-1,r}+(2k+2r-2i_0)b+\sum_{i=1}^{i_0-1}2\beta_i+(2k+1-2j_0)b+\sum_{j=1}^{j_0-1}2\gamma_j.$$

But the numbers $\beta_1, \beta_2, \dots, \beta_{i_0}, \gamma_1, \gamma_2, \dots, \gamma_{j_0}$ are exactly the elements of the sequence $\mathbf{z}(\Lambda)$ which are $\leqslant b$. So

$$i_0 + j_0 = \sum_{\substack{z \in \mathbf{z}(\Lambda) \\ z \leqslant b}} 1$$

and

$$\sum_{i=1}^{i_0-1} \beta_i + \sum_{j=1}^{j_0-1} \gamma_j = \sum_{\substack{z \in \mathbf{z}(\Lambda) \\ z < b}} z.$$

This shows (♡). ■

Now, the family of $\Lambda[b]$ depends only on the family of Λ (and not on Λ itself): indeed, $\mathbf{z}(\Lambda[b])$ is obtained from $\mathbf{z}(\Lambda)$ by removing the two entries equal to b. We will denote by $\mathscr{F}[b]$ the family of $\Lambda[b]$. Moreover, $Z_{\mathscr{F}[b]} = Z_{\mathscr{F}}$ and the map $\Lambda \mapsto \Lambda[b]$ induces a bijection between $\mathscr{F}$ and $\mathscr{F}[b]$, and also induces a bijection between $\mathscr{F}_\iota$ and $\mathscr{F}[b]_\iota$.

Moreover, the formula (♡) shows that the difference between $\mathbf{b}(\Lambda)$ and $\mathbf{b}(\Lambda[b])$ depends only on b and $\mathscr{F}$, so proving Lemma 6 for the pair $(\mathscr{F}, \iota)$ is equivalent to proving Lemma 6 for the pair $(\mathscr{F}[b], \iota)$. By applying several times this principle if necessary, this means that we may, and we will, assume that

$$X_{\mathscr{F}} = \varnothing.$$

1.H. Proof of Theorem 5(b). — We denote by $f_r < \cdots < f_1$ the elements of $Z_{\mathscr{F}}$ which are fixed by ι. We also set $f_{r+1} = 0$ and $f_0 = \infty$. As ι is r-admissible, the set $Z_{\mathscr{F}}^{(d)} = \{z \in Z_{\mathscr{F}} \mid f_{d+1} < z < f_d\}$ is ι-stable and contains no ι-fixed point (for $d \in \{0, 1, \dots, r\}$). Let $k_d = |Z_{\mathscr{F}}^{(d)}|/2$ and let ι_d be the restriction of ι to $Z_{\mathscr{F}}^{(d)}$. Then ι_d is a 0-admissible involution of $Z_{\mathscr{F}}^{(d)}$.

If $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \in \mathscr{F}_\iota$, we set $\beta^{(d)} = \beta \cap Z_{\mathscr{F}}^{(d)}$, $\gamma^{(d)} = \gamma \cap Z_{\mathscr{F}}^{(d)}$ and $\Lambda^{(d)} = \begin{pmatrix} \beta^{(d)} \\ \gamma^{(d)} \end{pmatrix}$. Then $\Lambda^{(d)} \in \mathbf{Sym}_{k_d}(0)$ and, if $\mathscr{F}^{(d)}$ denotes the family of $\Lambda^{(d)}$, then $\Lambda^{(d)} \in \mathscr{F}_{\iota_d}^{(d)}$.

Now, if $\Lambda' = \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix} \in \mathbf{Sym}_{k'}(0)$, we set

$$\mathbf{b}_d(\Lambda') = \sum_{i=1}^{k'} (2k' + 2d - 2i)\beta'_i + \sum_{j=1}^{k'} (2k' + 1 - 2j)\gamma'_j.$$

The number $\mathbf{b}_d(\Lambda')$ is called the $\mathbf{b}_d$-*invariant* of Λ'. It then follows from the definition of $\mathbf{b}$ and $\nabla_{k,r}$ that

$$(\spadesuit) \qquad \mathbf{b}(\Lambda) = \sum_{d=0}^{r} \mathbf{b}_d(\Lambda^{(d)}) - \nabla_{k,r} + \sum_{d=1}^{r} 2(k_0 + k_1 + \cdots + k_{d-1})\Big(f_d + \sum_{z \in Z^{(d)}} z\Big).$$

Since the map

$$\begin{array}{ccc} \mathscr{F}_\iota & \longrightarrow & \prod_{d=0}^r \mathscr{F}_{\iota_d}^{(d)} \\ \Lambda & \longmapsto & (\Lambda^{(0)}, \Lambda^{(1)}, \dots, \Lambda^{(d)}) \end{array}$$

is bijective and since $\mathbf{b}(\Lambda) - \sum_{d=0}^r \mathbf{b}_d(\Lambda^{(d)})$ depends only on $(\mathscr{F}, \iota)$ and not on Λ (as shown by the formula (♠)), Theorem 5(b) will follow from the following lemma :

Lemma 6. *There exists a unique symbol in $\mathscr{F}_{\iota_d}^{(d)}$ with minimal $\mathbf{b}_d$-invariant.*

The proof of Lemma 6 will be given in the next section.

2. Minimal $\mathbf{b}_d$-invariant

For simplifying notation, we set $Z = Z_{\mathscr{F}}^{(d)}$, $l = k_d$, $\mathscr{G} = \mathscr{F}^{(d)}$ and $\jmath = \iota_d$. Let us write $Z = \{z_1, z_2, \dots, z_{2l}\}$ with $z_1 < z_2 < \cdots < z_{2l}$. Recall from the previous secion that $\jmath$ is a 0-admissible involution of Z.

2.A. Construction. — We shall define by induction on $l \geqslant 0$ a symbol $\Lambda_\jmath^{(d)}(Z) \in \mathscr{G}_\jmath$. If $l = 0$, then $\Lambda_\jmath^{(d)}(Z)$ is obviously empty. So assume now that, for any set of non-zero integers Z' of order $2(l-1)$, for any 0-admissible involution $\jmath'$ of Z' and any $d' \geqslant 0$, we have defined a symbol $\Lambda_{\jmath'}^{(d')}(Z')$. Then $\Lambda_\jmath^{(d)}(Z) = \begin{pmatrix} \beta_\jmath^{(d)}(Z) \\ \gamma_\jmath^{(d)}(Z) \end{pmatrix}$ is defined as follows: let $Z' = Z \setminus \{z_1, \iota(z_1)\}$, $\jmath'$ the restriction of $\jmath$ to Z' and let

$$d' = \begin{cases} d-1 & \text{if } d \geqslant 1, \\ 1 & \text{if } d = 0. \end{cases}$$

Then $|Z'| = 2(l-1)$ and $\jmath'$ is 0-admissible. So $\Lambda_{\jmath'}^{(d')}(Z') = \begin{pmatrix} \beta_{\jmath'}^{(d')}(Z') \\ \gamma_{\jmath'}^{(d')}(Z') \end{pmatrix}$ is well-defined by the induction hypothesis. We then set

$$\beta_\jmath^{(d)}(Z) = \begin{cases} \beta_{\jmath'}^{(d')}(Z') \cup \{z_1\} & \text{if } d \geqslant 1, \\ \beta_{\jmath'}^{(d')}(Z') \cup \{\jmath(z_1)\} & \text{if } d = 0, \end{cases}$$

and

$$\gamma_\jmath^{(d)}(Z) = \begin{cases} \gamma_{\jmath'}^{(d')}(Z') \cup \{\jmath(z_1)\} & \text{if } d \geqslant 1, \\ \gamma_{\jmath'}^{(d')}(Z') \cup \{z_1\} & \text{if } d = 0. \end{cases}$$

Then Lemma 6 is implied by the next lemma :

Lemma 6⁺. *Let $\Lambda \in \mathscr{G}_\jmath$. Then $\mathbf{b}_d(\Lambda) \geqslant \mathbf{b}_d(\Lambda_\jmath^{(d)}(Z))$ with equality if and only if $\Lambda = \Lambda_\jmath^{(d)}(Z)$.*

The rest of this section is devoted to the proof of Lemma 6^+. We will first prove Lemma 6^+ whenever $d \in \{0,1\}$ using Lusztig's Theorem. We will then turn to the general case, which will be handled by induction on $l = |Z|/2$. We fix $\Lambda = \binom{\beta}{\gamma} \in \mathscr{G}_\imath$.

2.B. Proof of Lemma 6^+ whenever $d=1$. — Let z be a natural number strictly bigger than all the elements of Z. Let $\tilde{\Lambda} = \binom{\beta \cup \{z\}}{\gamma} \in \mathbf{Sym}_k(1)$. Then $\mathbf{b}_1(\Lambda) = \mathbf{b}(\tilde{\Lambda}) + C$, where C depends only on Z. Let $\tilde{\Lambda}_0 = \binom{z_1, z_3, \dots, z_{2l-1}, z}{z_2, \dots, z_{2l}}$. Since $\jmath$ is 0-admissible, it is easily seen that, if $\jmath(z_i) = z_j$, then $j - i$ is odd. So $\tilde{\Lambda}_0 \in \mathscr{G}_\jmath$. But, by **[Lu1**, §5], $\mathbf{b}(\tilde{\Lambda}) \geqslant \mathbf{b}(\tilde{\Lambda}_0)$ with equality if and only if $\tilde{\Lambda} = \tilde{\Lambda}_0$. So it is sufficient to notice that $\widetilde{\Lambda_\jmath^{(1)}(Z)} = \tilde{\Lambda}_0$, which is easily checked.

2.C. Proof of Lemma 6^+ whenever $d=0$. — Assume in this subsection, and only in this subsection, that $d = 0$ or 1. We denote by $\Lambda^{\mathrm{op}} = \binom{\gamma}{\beta} \in \mathscr{G}_\jmath$. It is readily seen from the construction that $\Lambda_\jmath^{(0)}(Z)^{\mathrm{op}} = \Lambda_\jmath^{(1)}(Z)$ and that

$$\mathbf{b}_1(\Lambda) = \mathbf{b}_0(\Lambda^{\mathrm{op}}) + \sum_{z \in Z} z.$$

So Lemma 6^+ for $d=0$ follows from Lemma 6^+ for $d=1$.

2.D. Proof of Lemma 6^+ whenever $d \geqslant 2$. — Assume now, and until the end of this section, that $d \geqslant 2$. We shall prove Lemma 6^+ by induction on $l = |Z|/2$. The result is obvious if $l = 0$, as well as if $l = 1$. So we assume that $l \geqslant 2$ and that Lemma 6^+ holds for $l' \leqslant l-1$. Write $\jmath(z_1) = z_{2m}$, where $m \leqslant l$ (note that $\jmath(z_1) \notin \{z_1, z_3, z_5, \dots, z_{2l-1}\}$ since $\jmath$ is 0-admissible).

Assume first that $m < l$. Then Z can we written as the union $Z = Z^+ \dot\cup Z^-$, where $Z^+ = \{z_1, z_2, \dots, z_{2m}\}$ and $Z^- = \{z_{2m+1}, z_{2m+2}, \dots, z_{2l}\}$ are $\jmath$-stable (since $\jmath$ is 0-admissible). If $\varepsilon \in \{+,-\}$, let $\jmath^\varepsilon$ denote the restriction of $\jmath$ to Z^ε, let $\beta^\varepsilon = \beta \cap Z^\varepsilon$, $\gamma^\varepsilon = \gamma \cap Z^\varepsilon$ and $\Lambda^\varepsilon = \binom{\beta^\varepsilon}{\gamma^\varepsilon}$, and let $\mathscr{G}^\varepsilon$ denote the family of Λ^ε. Then it is easily seen that $\Lambda^\varepsilon \in \mathscr{G}^\varepsilon_{\jmath^\varepsilon}$, that $\mathbf{b}_d(\Lambda) - \big(\mathbf{b}_d(\Lambda^+) + \mathbf{b}_d(\Lambda^-)\big)$ depends only on $(\mathscr{G}, \jmath)$ and that $\Lambda_\jmath^{(d)}(Z)^\varepsilon = \Lambda_{\jmath^\varepsilon}^{(d)}(Z^\varepsilon)$. By the induction hypothesis, $\mathbf{b}_d(\Lambda^\varepsilon) \geqslant \mathbf{b}_d(\Lambda_{\jmath^\varepsilon}^{(d)}(Z^\varepsilon))$ with equality if and only if $\Lambda^\varepsilon = \Lambda_{\jmath^\varepsilon}^{(d)}(Z^\varepsilon)$. So the result follows in this case. This means that we may, and we will, work under the following hypothesis:

Hypothesis. *From now on, and until the end of this section, we assume that* $\jmath(z_1) = z_{2l}$.

As in the construction of $\Lambda_J^{(d)}(Z)$, let $Z' = Z \setminus \{z_1, z_{2l}\} = \{z_2, z_3, \dots, z_{2l-1}\}$, let J' denote the restriction of J to Z' and let

$$d' = \begin{cases} d-1 & \text{if } d \geqslant 1, \\ 1 & \text{if } d = 0. \end{cases}$$

Then $|Z'| = 2(l-1)$ and J' is 0-admissible. Let $\Lambda' = \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix}$ where $\beta' = \beta \setminus \{z_1, z_{2l}\}$ and $\gamma' = \gamma \setminus \{z_1, z_{2l}\}$. Since $d \geqslant 2$, we have $z_1 \in \beta_J^{(d)}(Z)$ and $z_{2l} \in \gamma_J^{(d)}(Z)$. This implies that

$$(\bigstar) \qquad \mathbf{b}_d(\Lambda_J^{(d)}(Z)) = \mathbf{b}_{d-1}(\Lambda_{J'}^{(d-1)}(Z')) + z_{2l} + 2(l+d)z_1 + 2\sum_{z \in Z'} z.$$

If $z_1 \in \beta$, then $\Lambda = \Lambda_J^{(d)}(Z)$ if and only if $\Lambda' = \Lambda_{J'}^{(d')}(Z')$ and again

$$\mathbf{b}_d(\Lambda) = \mathbf{b}_{d-1}(\Lambda') + z_{2l} + 2(l+d)z_1 + 2\sum_{z \in Z'} z.$$

So the result follows from ($\bigstar$) and from the induction hypothesis.

This means that we may, and we will, assume that $z_1 \in \gamma$. In this case,

$$\mathbf{b}_d(\Lambda) = \mathbf{b}_{d+1}(\Lambda') + 2d z_{2l} + (2l+1)z_1.$$

Then it follows from ($\bigstar$) that

$$\mathbf{b}_d(\Lambda) - \mathbf{b}_d(\Lambda_J^{(d)}(Z)) = \mathbf{b}_{d+1}(\Lambda') - \mathbf{b}_{d-1}(\Lambda_{J'}^{(d-1)}(Z')) + (2d-1)(z_{2l} - z_1) - 2\sum_{z \in Z'} z.$$

So, by the induction hypothesis,

$$\mathbf{b}_d(\Lambda) - \mathbf{b}_d(\Lambda_J^{(d)}(Z)) \geqslant \mathbf{b}_{d+1}(\Lambda_{J'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{J'}^{(d-1)}(Z')) + (2d-1)(z_{2l} - z_1) - 2\sum_{z \in Z'} z.$$

Since $z_{2l} - z_1 > z_{2l-1} - z_2$, it is sufficient to show that

$$(?) \qquad \mathbf{b}_{d+1}(\Lambda_{J'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{J'}^{(d-1)}(Z')) \geqslant -(2d-1)(z_{2l-1} - z_2) + 2\sum_{z \in Z'} z.$$

This will be proved by induction on the size of Z'. First, if $J(z_2) < z_{2l}$, then we can separate Z' into two J'-stable subsets and a similar argument as before allows to conclude thanks to the induction hypothesis.

So we assume that $J'(z_2) = z_{2l-1}$. Let $Z'' = Z' \setminus \{z_2, z_{2l-1}\}$ and let J'' denote the restriction of J' to Z''. Since $z_2 \in \beta_{J'}^{(d+1)}(Z')$, we can apply ($\bigstar$) one step further to get

$$\begin{aligned} \mathbf{b}_{d+1}(\Lambda_{J'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{J'}^{(d-1)}(Z')) &= \mathbf{b}_d(\Lambda_{J''}^{(d)}(Z'') + z_{2l-1} + 2(l+d)z_2 + 2\sum_{z \in Z''} z \\ &\quad -\big(\mathbf{b}_{d-2}(\Lambda_{J''}^{(d-2)}(Z'')) + z_{2l-1} + 2(l+d-2)z_2 + 2\sum_{z \in Z''} z\big) \\ &= \mathbf{b}_d(\Lambda_{J''}^{(d)}(Z'') - \mathbf{b}_{d-2}(\Lambda_{J''}^{(d-2)}(Z'')) + 4z_2. \end{aligned}$$

So, by the induction hypothesis,

$$\begin{aligned} \mathbf{b}_{d+1}(\Lambda_{J'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{J'}^{(d-1)}(Z')) &\geqslant -(2d-3)(z_{2l-2}-z_3)+2\sum_{z\in Z''} z + 4z_2 \\ &\geqslant -(2d-3)(z_{2l-1}-z_2)+2\sum_{z\in Z'} z + 2z_2 - 2z_{2l-1} \\ &= -(2d-1)(z_{2l-1}-z_2)+2\sum_{z\in Z'} z, \end{aligned}$$

as desired. This shows (?) and completes the proof of Lemma 6^+.

3. Complex reflection groups

If $\mathscr{W}$ is a complex reflection group, then R. Rouquier and the author have also defined Calogero-Moser cellular characters and Calogero-Moser families (see [**BoRo1**] or [**BoRo2**]). If $\mathscr{W}$ is of type $G(l,1,n)$ (in Shephard-Todd classification), then Leclerc and Miyachi [**LeMi**, §6.3] proposed, in link with canonical bases of $U_v(\mathfrak{sl}_\infty)$-modules, a family of characters that could be good analogue of constructible characters: let us call them the *Leclerc-Miyachi constructible characters* of $G(l,1,n)$. If $l=2$, then they coincide with constructible characters [**LeMi**, Theorem 10].

Of course, it would be interesting to know if Calogero-Moser cellular characters coincide with the Leclerc-Miyachi ones: this seems rather complicated but it should be at least possible to check if the Leclerc-Miyachi constructible characters satisfy the analogous properties with respect to the b-invariant.

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CÉDRIC BONNAFÉ, Institut de Mathématiques et de Modélisation de Montpellier (CNRS: UMR 5149), Université Montpellier 2, Case Courrier 051, Place Eugène Bataillon, 34095 MONTPELLIER Cedex, FRANCE • *E-mail :* `cedric.bonnafe@univ-montp2.fr`